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ON THE PERTURBABILITY OF THE ASYMPTOTIC MANIFOLD OF A PERTURBED SYSTEM OF DIFFERENTIAL EQUATIONS

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ABSTRACT. We investigate the asymptotic relationship between the solutions of a linear differential system and its perturbed system. Our results depend upon a known result of F. Brauer on asymptotic equilibrium. We also study the asymptotic manifold of solutions of the nonlinear system generated by the solutions of the corresponding linear system.

1. Recently, Brauer and Wong [2] and Onuchic [5] obtained general results on the asymptotic relationship between the solutions of a linear differential system and its perturbed system. Toroželidze [6] considered the problem of perturbing the asymptotic manifold of a nonlinear scalar equation. Hallam and Heidel [3] extended the results of Toroželidze to nonlinear systems and also discussed the asymptotic relationships of solutions.

We, in the present paper, wish to investigate these problems further. Our results depend upon a known basic result [1] on asymptotic equilibrium and consequently the proofs are short and fundamental. Our approach has thrown considerable light on the problems and at the same time improved the results of [2] and [3] by weakening the hypotheses. This is explained in detail in a remark at the end of the paper.

2. Let J denote the half-line $0 \leqq t < \infty$ and R^n, the euclidean n-space. Let $\|\cdot\|$ denote any convenient norm of a vector and the corresponding norm of a matrix. We consider the nonlinear differential system

$$x' = A(t)x + f(t, x), \quad x(t_0) = x_0, \qquad t_0 \in J, \tag{2.1}$$

where $A(t)$ is a continuous $n \times n$ matrix for $t \in J$ and $f \in C[J \times R^n, R^n]$.

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Suppose that $Y(t, t_0)$ is the fundamental matrix solution of the corresponding linear system

$$y' = A(t)y, \qquad y(t_0) = y_0, \tag{2.2}$$

such that $Y(t_0, t_0) = \mathrm{I}$ (unit matrix). Then, the transformation

$$x = Y(t, t_0)z \tag{2.3}$$

reduces (2.1) to the system

$$z' = Y^{-1}(t, t_0)f(t, Y(t, t_0)z) \equiv F(t, z), \qquad z(t_0) = x_0, \tag{2.4}$$

which plays an important role in our discussion below.

A known result of F. Brauer [1] will now be stated in a modified form whose proof requires no significant changes.

THEOREM 0. *Assume that*

(i) $F \in C[J \times R^n, R^n]$ *and* $\|F(t, z)\| \leqq g(t, \|z\|)$, $(t, z) \in J \times R^n$;

(ii) $g \in C[J \times R^+, R^+]$ *and* $g(t, u)$ *is monotone nondecreasing in* u *for each* $t \in J$;

(iii) *for a given* $u_0 > 0$, *the maximal solution* $r(t) = r(t, t_0, u_0)$ *of the scalar differential equation*

$$u' = g(t, u), \qquad u(t_0) = u_0, \tag{2.5}$$

is bounded on $[t_0, \infty)$.

Then, the differential system (2.4) *has asymptotic equilibrium in the set* $A = [x \in R^n : \|x\| \leqq u_0]$.

This theorem and the techniques of its proof will be used frequently in the sequel.

Let us now introduce the following definitions. For that purpose, assume that $\Delta(t)$ is a continuous $n \times n$ matrix for $t \in J$, satisfying the estimate

$$\|\Delta(t)Y(t, t_0)\| \leqq \alpha(t), \qquad t \geqq t_0, \tag{2.6}$$

where $\alpha(t)$ is a continuous positive function for $t \in J$.

DEFINITION 1. The differential systems (2.1) and (2.2) are said to be generalized asymptotically equivalent in a set $A \subset R^n$, if

(i) given any $(t_0, x_0) \in J \times A$, there exists a $c \in R^n$ such that every solution $x(t, t_0, x_0)$ of (2.1) satisfies the order relation

$$\|\Delta(t)[x(t, t_0, x_0) - Y(t, t_0)c]\| = o(\alpha(t)), \qquad t \to \infty; \tag{2.7}$$

(ii) given any solution $y(t) = Y(t, t_0)c$ of (2.2) with $c \in A$, there exists a solution $x(t, t_0, x_0)$ of (2.1) on $[t_0, \infty)$ such that (2.7) holds.

DEFINITION 2. The differential systems (2.1) and (2.2) are said to be generalized eventually asymptotically equivalent, if given any

$x_0 \in R^n$, there exists a $T_0 = T_0(x_0) > 0$ and a $c \in R^n$ such that every solution $x(t, t_0, x_0)$, $t_0 \geqq T_0$, of (2.1) satisfies (2.7) and conversely, given any $c \in R^n$, there exists a $T_0 = T_0(c) > 0$ and a solution $x(t, t_0, x_0)$, $t_0 \geqq T_0$, of (2.1) satisfying (2.7).

Notice that whenever $\Delta(t) = Y^{-1}(t, t_0)$ and $\alpha(t) = 1$, the order relation (2.7) takes the form

$$\|z(t, t_0, x_0) - c\| = o(1), \qquad t \to \infty, \tag{2.8}$$

where $z(t, t_0, x_0)$ is a solution of (2.4). The foregoing definitions are then reduced to the corresponding asymptotic equivalence notions with respect to the systems (2.4) and $\xi' = 0$, or equivalently to the concepts of asymptotic equilibrium of the system (2.4).

DEFINITION 3. Given $\Delta(t)$, $\alpha(t)$, the set $S(\Delta, \alpha)$ consisting of all solutions $x(t, t_0, x_0)$, $(t_0, x_0) \in J \times R^n$, that verify the order relation (2.7) for some $c \in R^n$, is called the asymptotic manifold of (2.1) generated by (2.2).

We are also interested in the submanifold $S_0(\Delta, \alpha)$ defined by

$$S_0(\Delta, \alpha) = \{x(t, t_0, x_0) \in S(\Delta, \alpha) : Y^{-1}(t, t_0)x(t, t_0, x_0) \text{ is bounded on } [t_0, \infty)\}.$$

DEFINITION 4. A subset $\tilde{S}$ of $S(\Delta, \alpha)$ is said to be perturbable if given any solution $x_0(t, t_0, x_0)$ of (2.1) in $\tilde{S}$ there exists a $\delta > 0$ such that every solution $x_1(t, t_0, x_1)$ of (2.1) satisfying $\|x_0 - x_1\| < \delta$, belongs to $\tilde{S}$.

3. The following two results deal with the asymptotic relationship between the systems (2.1) and (2.2).

THEOREM 1. *Assume that*

(i) *the estimate* (2.6) *holds*;

(ii) *for* $(t, x) \in J \times R^n$,

$$\|Y^{-1}(t,t_0)f(t,x)\| \leqq g\left(t, \frac{\|\Delta(t)x\|}{\alpha(t)}\right) \tag{2.9}$$

where $g \in C[J \times R^+, R^+]$ *and* $g(t, u)$ *is nondecreasing in* u *for each* $t \in J$;

(iii) *the hypothesis* (iii) *of Theorem* 0 *is verified. Then, the differential systems* (2.1) *and* (2.2) *are generalized asymptotically equivalent in the set* $A = [x \in R^n : \|x\| \leqq u_0]$.

PROOF. In view of the relations (2.3), (2.4), (2.6), (2.9) and the monotonic character of $g(t, u)$ we obtain

$$\|F(t, z)\| = \|Y^{-1}(t, t_0)f(t, Y(t, t_0)z)\| \leqq g(t, \|z\|). \tag{2.10}$$

Hence, the differential system (2.4) verifies all the assumptions of Theorem 0 and consequently it has asymptotic equilibrium in the set A. All that is necessary to complete the proof is to notice the truth of the inequality

$$\begin{aligned} \|\Delta(t)[x(t, t_0, x_0) - Y(t, t_0)c]\| &\leqq \|\Delta(t)Y(t, t_0)\| \, \|z(t, t_0, x_0) - c\| \\ &\leqq \alpha(t)\|z(t, t_0, x_0) - c\|, \end{aligned} \tag{2.11}$$

which results using (2.3) and (2.6).

THEOREM 2. *Let the hypotheses* (i) *and* (ii) *of Theorem* 1 *hold. Suppose further*

(iii*) $\int^\infty g(s, M)ds < \infty$, *for each positive* M.

Then

(a) *the differential systems* (2.1) *and* (2.2) *are generalized eventually asymptotically equivalent*;

(b) *any solution* $x(t, t_0, x_0)$, $(t_0, x_0) \in J \times R^n$, *of* (2.1) *such that* $Y^{-1}(t, t_0)x(t, t_0, x_0)$ *is bounded on* $[t_0, \infty)$, *is a member of* $S(\Delta, \alpha)$.

The proof of this theorem depends on the following lemma. It is of interest in itself since it throws much light on the condition (iii*) by exhibiting the same in terms of equivalent and more convenient criteria.

LEMMA. *Assume that* $g \in C[J \times R^+, R^+]$ *and* $g(t, u)$ *is monotone nondecreasing in* u *for each* $t \in J$. *Then, the following are equivalent*:

(A) *given any* $u_0 > 0$, *there exists a* $T_0 = T_0(u_0) > 0$ *such that the maximal solution* $r(t, t_0, u_0)$, $t_0 \geqq T_0$, *of* (2.5) *is bounded on* $[t_0, \infty)$;

(B) *given any* $u^0 > 0$, *there is a solution* $u(t)$ *of* (2.5) *existing on some interval* $[t_1, \infty)$ *such that* $\lim_{t\to\infty} u(t) = u^0$;

(C) $\int^\infty g(s, M)ds < \infty$, *for each positive* M.

The proof of this lemma can be given along the lines of the proof of the lemma in [3] and therefore we shall omit it.

PROOF OF THEOREM 2. Since (iii*) implies (A) by lemma, the generalized eventual asymptotic equivalence of the systems (2.1) and (2.2) follows by adapting the proof of Theorem 0, together with the observations that lead to the inequalities (2.10) and (2.11). This proves (a).

To prove (b), let $x(t, t_0, \cdot_0)$ be a solution of (2.1) such that $Y^{-1}(t, t_0) \cdot x(t, t_0, x_0)$ is bounded on $[t_0, \infty)$. This implies, by (2.3) that, $z(t, t_0, x_0)$ is bounded on $[t_0, \infty)$, where $z(t, t_0, x_0)$ is a solution of (2.4). Suppose that $\|z(t, t_0, x_0)\| \leqq M$, $t \geqq t_0$. Since (iii*) implies (A), by lemma, given $M > 0$, there exists a $T_0 = T_0(M) > 0$ such that the maximal solution $r(t, T_0, M)$ of (2.5) is bounded on $[T_0, \infty)$. At $t = T_0$, we have

$\|z(T_0, t_0, x_0)\| \leqq M = r(T_0, T_0, M)$. Letting $z^*(t, T_0, z(T_0))$ be any solution of (2.4) through $(T_0, z(T_0))$, where $z(T_0) = z(T_0, t_0, x_0)$ and denoting by $m(t) = \|z^*(t, T_0, z(T_0))\|$, we obtain the differential inequality

$$D^+m(t) \leqq g(t, m(t)), \qquad t \geqq T_0.$$

It then follows by a well-known result (Theorem 1.4.1 in [4]) that

$$\|z^*(t, T_0, z(T_0))\| \leqq r(t, T_0, M), \qquad t \geqq T_0.$$

Since $z(t, t_0, x_0)$ is one of those solutions $z^*(t, T_0, z(T_0))$, the inequality

$$\|z(t, t_0, x_0)\| \leqq r(t, T_0, M), \qquad t \geqq T_0,$$

is also true. It is now easy to conclude that there exists a $c \in R^n$ such that $\lim_{t\to\infty} z(t, t_0, x_0) = c$, following the first part of the proof of Theorem 0. This, in turn, shows that (2.7) holds, because of (2.11). The proof is complete.

The theorem that follows gives sufficient conditions for the perturbability of the manifold $S_0(\Delta, \alpha)$.

THEOREM 3. *The assumptions of Theorem* 2, *together with the uniqueness and continuous dependence on initial conditions of solutions of* (2.1), *imply that* $S_0(\Delta, \alpha)$ *is perturbable.*

PROOF. Let $x_0(t) = x(t, t_0, x_0)$ be the solution of (2.1) such that $x_0(t) \in S_0(\Delta, \alpha)$. Then $Y^{-1}(t, t_0)x_0(t)$ is bounded on $[t_0, \infty)$. In view of (2.3), we deduce that $\|z_0(t)\| \leqq M_0$, $t \geqq t_0$, for some $M_0 > 0$, where $z_0(t) = z(t, t_0, x_0)$ is the solution of (2.4). Clearly, because of (2.11), it is enough to show the existence of a $\delta > 0$ such that $\|x_0 - x_1\| < \delta$ implies that any solution $z_1(t) = z(t, t_0, x_1)$ of (2.4) is bounded on $[t_0, \infty)$ and that $\lim_{t\to\infty} z_1(t) = c$. Since, by lemma, (iii*) implies A, given $M = M_0 + 1$, there exists a $T_0 = T(M) > 0$ such that the maximal solution $r(t) = r(t, T_0, M)$ of (2.5) is bounded on $[T_0, \infty)$. As $g(t, u) \geqq 0$, $r(t)$ is nondecreasing and consequently $\lim_{t\to\infty} r(t)$ exists. It then follows that $\|z_0(t)\| < r(t)$, $t \geqq T_0$. Let $\epsilon = \frac{1}{2}(M - \|z_0(T_0)\|) > 0$ be given. Then, by the assumed continuous dependence of solutions on initial values, there exists a $\delta > 0$ such that $\|x_0 - x_1\| < \delta$ implies $\|z_1(T_0) - z_0(T_0)\| < \epsilon$. Hence, we derive that $\|z_1(T_0)\| \leqq \|z_0(T_0)\| + \epsilon \leqq M$.

Using the uniqueness of solutions and the corresponding arguments as in the proof of Theorem 2, it is now easy to complete the proof.

THEOREM 4. *If, in addition to the hypotheses of Theorem* 3, *we assume that*

$$\| Y^{-1}(t, t_0)\Delta^{-1}(t)\| \leqq \beta(t), \qquad t \geqq t_0, \tag{2.12}$$

where $\beta(t)$ is a continuous positive function for $t \in J$ such that $\alpha(t)\beta(t) \leqq L$ on J, then $S(\Delta, \alpha)$ is perturbable.

PROOF. Since $S_0(\Delta, \alpha)$ is perturbable by Theorem 3, it suffices to show that $S_0(\Delta, \alpha) = S(\Delta, \alpha)$.

Let $x_0(t) \in S(\Delta, \alpha)$, where $x_0(t) = x(t, t_0, x_0)$ is the solution of (2.1). Then, by (2.7), there exists a constant $M_1 > 0$ satisfying $\|\Delta(t)x_0(t)\| \leqq M_1\alpha(t)$, $t \geqq t_0$.

Since

$$\begin{aligned} \| Y^{-1}(t, t_0)x_0(t)\| &= \| Y^{-1}(t, t_0)\Delta^{-1}(t)\Delta(t)x_0(t)\| \\ &\leqq \| Y^{-1}(t, t_0)\Delta^{-1}(t)\| \, \|\Delta(t)x_0(t)\|, \end{aligned}$$

it follows that $Y^{-1}(t, t_0)x_0(t)$ is bounded on $[t_0, \infty)$, which implies that $x_0(t) \in S_0(\Delta, \alpha)$. Because, by definition $S_0(\Delta, \alpha) \subset S(\Delta, \alpha)$, the theorem is proved.

In general, when comparison technique is used, a property of the scalar differential equation yields the corresponding property of the given differential system. A natural question therefore arises whether a similar situation occurs in the case of perturbability of manifolds. The following theorem answers this question in the affirmative.

Let $S(1, 1)$ denote the set of all positive solutions $u(t, t_0, u_0)$, $t_0 \in J$, $u_0 > 0$, of the scalar differential equation (2.5) satisfying the order relation $u(t, t_0, u_0) - \xi = o(1)$, $t \to \infty$. Then, we have

THEOREM 5. *Let the assumptions* (i) *and* (ii) *of Theorem* 1 *hold. Suppose further that the uniqueness and continuous dependence on initial values of solutions of* (2.1) *and* (2.5) *are assured. Then the perturbability of $S(1, 1)$ implies the perturbability of $S_0(\Delta, \alpha)$.*

PROOF. Suppose $S(1, 1)$ is perturbable. If all solutions $u(t, t_0, u_0)$ of (2.5) are bounded, the assertion clearly follows. Let $u(t, t_0, \beta)$ be the bounded solutions and $u(t, t_0, \gamma)$ be the unbounded solutions. Let $\beta^* = \sup \beta$ and $\gamma^* = \inf \gamma$. Evidently $\beta^* = \gamma^*$ and $u(t, t_0, \beta^*)$ is the solution of (2.5). Since $S(1, 1)$ is perturbable, it is easy to see that $u(t, t_0, \beta^*)$ is unbounded. Notice that the maximal interval of existence of $u(t, t_0, \beta^*)$ may be $[t_0, \sigma)$, where $\sigma \leqq \infty$. In any case, by continuous dependence of solutions on initial values and the fact that $u(t, t_0, \beta^*)$ is unbounded, given any $u_0 > 0$, it is possible to find a $T_0 = T(u_0) > 0$ such that the solution $u(t, T_0, u_0)$ is bounded on $[T_0, \infty)$. Hence the condition (A) is true and by Theorem 3, the conclusion follows.

REMARKS. A number of remarks are now in order. First of all, the

proofs of some earlier results have been shortened and clarified. Theorems 1 and 3 of Brauer and Wong [2] are included in our Theorem 1. Notice that their proof of Theorem 1 requires the additional hypothesis of uniqueness of solutions of the scalar equation (2.5) to work and the extra assumptions of their Theorem 3 become superfluous.

Theorem 2 of the present paper generalizes a similar result of Onuchic [5]. Taking $\Delta(t) = Y^{-1}(t, t_0)$, $\alpha(t) \equiv 1$ and $h_M(t) = g(t, M)$, we see that the hypothesis (H) of Onuchic is verified whenever $\|z\| \leqq M$. Observe also that the conclusion of our result is slightly more than that of Onuchic's (refer Theorems 1 and 2 in [5]), since, even without the boundedness of $Y^{-1}(t, t_0)x(t)$, we obtain generalized eventual asymptotic equivalence of the systems.

At this point, we become aware of the work of Hallam and Heidel [3] on the perturbability of the asymptotic manifolds. We pointed out to Hallam that their main result (refer Theorem 4 in [3]) requires additional hypotheses to fix it up. We considered the same problems and our results (Theorems 3 and 4) significantly weaken their hypotheses even after the fix-up which Hallam communicated to us. Namely, we do not require the uniqueness of solutions of (2.5) and the monotonicity condition of $g(t, u)/u$ in u. Our Theorem 5 once again establishes the force of the general comparison technique. It is interesting to note that, instead of using (B) as Hallam and Heidel have done, we have employed the equivalent condition (A) which has facilitated in reducing the results to Theorem 0 and consequently in weakening the assumptions.

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