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ON A CONJECTURE OF A. J. HOFFMAN

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ABSTRACT. A 3-polytope P and four closed convex sets $C_1, \cdots, C_4$ in P are described, having the following property: every line which meets P meets at least one of the C_i's, and for every collection of polytopes $D_1, \cdots, D_4$, with $D_i \subseteq C_i$ for all $1 \leqq i \leqq 4$, there exists a line which meets P and misses all of the D_i's. This is a counterexample to a conjecture of A. J. Hoffman.

The following question was recently raised by A. J. Hoffman [2]: "If P is a d-polytope, $t>0$ an integer and $C_1, \cdots, C_k$ closed convex sets in P, such that every t-flat that meets P meets $\bigcup_{i=1}^{k} C_i$; do there exist polytopes $D_1, \cdots, D_k$, with $D_i \subseteq C_i$ for all $1 \leqq i \leqq k$, such that if a t-flat meets P, it meets $\bigcup_{i=1}^{k} D_i$?"

The purpose of this note is to give a counterexample to this conjecture, with $d=3$, $t=1$ and $k=4$ (see Remarks 1 and 2).

A *d-polytope* here means the convex hull of a set of finitely many points in the Euclidean d-space E^d, having a nonempty interior, see [1].

Let P be the polytope on the following six vertices in E^3: $A_1 = (0, 0, 0)$, $A_2 = (3, 0, 0)$, $A_3 = (0, 3, 0)$, $A_4 = (0, 0, 20)$, $A_5 = (3, 0, 20)$ and $A_6 = (0, 3, 20)$.

P is a prism with a base B_1 at the level $z=0$ and a base B_2 at the level $z=20$, where

$$B_1 = \text{convex hull}\{A_1, A_2, A_3\} \text{ and } B_2 = \text{convex hull}\{A_4, A_5, A_6\}.$$

Let L_1 (L_2) be the line passing through the points (0, 0, 1) and (1, 0, 1), ((0, 0, 3) and (0, 1, 3), resp.). Let T_1 (T_2) be the cylinder containing all points within a distance of $\leqq 1$ to L_1 (L_2, resp.).

We define C_i, $1 \leqq i \leqq 3$, as follows

$$C_1 = \text{convex hull}[(T_1 \cap P) \cup \{A_1, A_2, A_4, A_5\}],$$
$$C_2 = \text{convex hull}[(T_2 \cap P) \cup \{A_1, A_3, A_4, A_6\}],$$
$$C_3 = \text{convex hull}\{A_2, A_3, A_5, A_6\}.$$

C_4 is defined as a subset of B_1, consisting of all the points M in B_1,

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for which there exists a point N in B_2, such that the segment MN has distance $\geqq 1$ to both of the two lines L_1 and L_2.[1]

$C_1, \cdots, C_4$ were so chosen as to assure that if a line meets P, it meets $\bigcup_{i=1}^4 C_i$.

To justify our claim, it will suffice to prove the following

LEMMA 1. *C_4 is a closed convex set which is not a polytope.*

LEMMA 2. *If D_i is a polytope in C_i, for all $1 \leqq i \leqq 4$, then there exists a line E, such that $E \cap P \neq \emptyset$ and $E \cap (\bigcup_{i=1}^4 D_i) = \emptyset$.*

Proof of Lemma 1. If $M=(x, y, 0)$ is a point in C_4, then for some point N of B_2 the segment MN does not meet T_1 and T_2, or else it is tangent to one, or both of them. We observe that if a point N exists, then another such point can be chosen to be a point of the edge A_5A_6, hence of the form $(3\lambda, 3(1-\lambda), 20)$ for some $0 \leqq \lambda \leqq 1$.

Let α be a subset of the positive orthant of the xy-plane, consisting of all the points $M=(x, y, 0)$, for which there exists a value of λ, $0 \leqq \lambda \leqq 1$, such that the segment MN, where $N=(3\lambda, 3(1-\lambda), 20)$, is tangent to *both* T_1 and T_2.

Since MN is tangent to T_1, it follows that

$$9y^2 - 3\lambda y + 3y = 10 \tag{1}$$

(by setting the distance of the line passing through MN to the line L_1 be equal to 1).

Similarly, since MN is tangent to T_2, it follows that

$$36x^2 + 39x\lambda + 9\lambda^2 = 50. \tag{2}$$

Let (x_λ, y_λ) be the positive solution of the equations (1) and (2), for every $0 \leqq \lambda \leqq 1$. It follows easily that $x_\lambda \geqq x_1$ and $y_\lambda \geqq y_0$ for all $0 \leqq \lambda \leqq 1$.

Moreover, the boundary of C_4 is the union of the following three segments: $\{(x, y_0) \mid x_0 \leqq x \leqq 3-y_0\}$, $\{(x_1, y) \mid y_1 \leqq y \leqq 3-x_1\}$ and $\{(x, y) \mid x \geqq x_1, y \geqq y_0 \text{ and } x+y=3\}$, together with the arc α, represented parametrically and implicitly by (1) and (2), with $0 \leqq \lambda \leqq 1$.

To show that C_4 is convex, it is obviously enough to prove that α is a convex curve, i.e., that $d^2y/dx^2 > 0$ at every point of α.

This is a straightforward computation, which we outline: (1) implies (3) and (4), while (2) implies (5) and (6), where:

$$\frac{dx}{d\lambda} = -\frac{13x + 6\lambda}{24x + 13\lambda}, \tag{3}$$

[1] Suppose there is a candle at each point of B_2, and every ray passing through the interior of T_1 and T_2 is absorbed; C_4 is then the illuminated part of B_1.

$$\frac{d^2x}{d\lambda^2} = \frac{25x - 25\lambda(dx/d\lambda)}{(24x + 13\lambda)^2}, \tag{4}$$

$$\frac{dy}{d\lambda} = \frac{y}{6y - \lambda + 1}, \tag{5}$$

$$\frac{d^2y}{d\lambda^2} = \frac{y + (1 - \lambda)dy/d\lambda}{(6y - \lambda + 1)^2}. \tag{6}$$

Since $x > 0$, $y > 0$ and $0 \leqq \lambda \leqq 1$, it follows from (3) that $dx/d\lambda < 0$, and from (5) that $dy/d\lambda > 0$; these, in turn, imply that $d^2x/d\lambda^2 > 0$ and $d^2y/d\lambda^2 > 0$.

Using the well-known formula

$$\frac{d^2y}{dx^2} = \left(\frac{dx}{d\lambda}\cdot\frac{d^2y}{d\lambda^2} - \frac{dy}{d\lambda}\cdot\frac{d^2x}{d\lambda^2}\right) \Big/ \left(\frac{dx}{d\lambda}\right)^3$$

it follows immediately that $d^2y/dx^2 > 0$ on α. Moreover, $d^2y/dx^2 > 0$ implies that α is not a segment.

As a result, C_4 is a closed convex set which is not a polytope, and the proof of Lemma 1 is completed.

Proof of Lemma 2. Let $D_1, \cdots, D_4$ be polytopes, with $D_i \subseteq C_i$ for all $1 \leqq i \leqq 4$.

D_4 is a convex (planar) polygon in C_4, and C_4 is, by Lemma 1, a closed convex (planar) set which is not a polygon. Therefore, there exists a point R on the arc α, such that $R \notin D_4$. Since D_4 is convex, there exists the point R^*, nearest to R in D_4. The point $S = \frac{1}{2}(R + R^*)$ is an interior point of C_4 and $S \notin D_4$.

Let S' be a point in the upper base B_2 of P, such that the segment SS' is of distance $\geqq 1$ from the two lines L_1 and L_2. Since S belongs to the interior of C_4, we may assume, without loss of truth, that S' belongs to the interior of B_2.

Let S_1S_1' be a segment, obtained from SS' by an ϵ-push towards the face C_3 of P, such that $S_1S_1' \| SS'$, S_1 is an interior point of $C_4 - D_4$, S_1' is an interior point of B_2 and S_1S_1' is of distance > 1 from both L_1 and L_2.

The line E, containing the segment S_1S_1', is such that $E \cap P \neq \emptyset$ and $E \cap (\bigcup_{i=1}^4 D_i) = \emptyset$.

This completes the proof of Lemma 2.

REMARK 1. The original example, as presented to the Society, involved $k = 10$ (with $d = 3$ and $t = 1$); a remark by (a student of) A. J. Hoffman reduced it to $k = 6$; that remark enabled us to further reduce the value of k to 4.

REMARK 2. By taking the Cartesian product of our example with the q-dimensional cube I^q, we derive additional counterexamples in the cases where $d=q+3$ and $t=q+1$, for all $q\geqq 1$.

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