

## ON A CONJECTURE OF A. J. HOFFMAN

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**ABSTRACT.** A 3-polytope  $P$  and four closed convex sets  $C_1, \dots, C_4$  in  $P$  are described, having the following property: every line which meets  $P$  meets at least one of the  $C_i$ 's, and for every collection of polytopes  $D_1, \dots, D_4$ , with  $D_i \subseteq C_i$  for all  $1 \leq i \leq 4$ , there exists a line which meets  $P$  and misses all of the  $D_i$ 's. This is a counterexample to a conjecture of A. J. Hoffman.

The following question was recently raised by A. J. Hoffman [2]: "If  $P$  is a  $d$ -polytope,  $t > 0$  an integer and  $C_1, \dots, C_k$  closed convex sets in  $P$ , such that every  $t$ -flat that meets  $P$  meets  $\bigcup_{i=1}^k C_i$ ; do there exist polytopes  $D_1, \dots, D_k$ , with  $D_i \subseteq C_i$  for all  $1 \leq i \leq k$ , such that if a  $t$ -flat meets  $P$ , it meets  $\bigcup_{i=1}^k D_i$ ?"

The purpose of this note is to give a counterexample to this conjecture, with  $d = 3$ ,  $t = 1$  and  $k = 4$  (see Remarks 1 and 2).

A  $d$ -polytope here means the convex hull of a set of finitely many points in the Euclidean  $d$ -space  $E^d$ , having a nonempty interior, see [1].

Let  $P$  be the polytope on the following six vertices in  $E^3$ :  $A_1 = (0, 0, 0)$ ,  $A_2 = (3, 0, 0)$ ,  $A_3 = (0, 3, 0)$ ,  $A_4 = (0, 0, 20)$ ,  $A_5 = (3, 0, 20)$  and  $A_6 = (0, 3, 20)$ .

$P$  is a prism with a base  $B_1$  at the level  $z = 0$  and a base  $B_2$  at the level  $z = 20$ , where

$$B_1 = \text{convex hull} \{A_1, A_2, A_3\} \text{ and } B_2 = \text{convex hull} \{A_4, A_5, A_6\}.$$

Let  $L_1$  ( $L_2$ ) be the line passing through the points  $(0, 0, 1)$  and  $(1, 0, 1)$ ,  $((0, 0, 3)$  and  $(0, 1, 3)$ , resp.). Let  $T_1$  ( $T_2$ ) be the cylinder containing all points within a distance of  $\leq 1$  to  $L_1$  ( $L_2$ , resp.).

We define  $C_i$ ,  $1 \leq i \leq 3$ , as follows

$$C_1 = \text{convex hull}[(T_1 \cap P) \cup \{A_1, A_2, A_4, A_5\}],$$

$$C_2 = \text{convex hull}[(T_2 \cap P) \cup \{A_1, A_3, A_4, A_6\}],$$

$$C_3 = \text{convex hull}\{A_2, A_3, A_5, A_6\}.$$

$C_4$  is defined as a subset of  $B_1$ , consisting of all the points  $M$  in  $B_1$ ,

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for which there exists a point  $N$  in  $B_2$ , such that the segment  $MN$  has distance  $\geq 1$  to both of the two lines  $L_1$  and  $L_2$ .<sup>1</sup>

$C_1, \dots, C_4$  were so chosen as to assure that if a line meets  $P$ , it meets  $\bigcup_{i=1}^4 C_i$ .

To justify our claim, it will suffice to prove the following

**LEMMA 1.**  $C_4$  is a closed convex set which is not a polytope.

**LEMMA 2.** If  $D_i$  is a polytope in  $C_i$ , for all  $1 \leq i \leq 4$ , then there exists a line  $E$ , such that  $E \cap P \neq \emptyset$  and  $E \cap (\bigcup_{i=1}^4 D_i) = \emptyset$ .

**Proof of Lemma 1.** If  $M = (x, y, 0)$  is a point in  $C_4$ , then for some point  $N$  of  $B_2$  the segment  $MN$  does not meet  $T_1$  and  $T_2$ , or else it is tangent to one, or both of them. We observe that if a point  $N$  exists, then another such point can be chosen to be a point of the edge  $A_5A_6$ , hence of the form  $(3\lambda, 3(1-\lambda), 20)$  for some  $0 \leq \lambda \leq 1$ .

Let  $\alpha$  be a subset of the positive orthant of the  $xy$ -plane, consisting of all the points  $M = (x, y, 0)$ , for which there exists a value of  $\lambda$ ,  $0 \leq \lambda \leq 1$ , such that the segment  $MN$ , where  $N = (3\lambda, 3(1-\lambda), 20)$ , is tangent to both  $T_1$  and  $T_2$ .

Since  $MN$  is tangent to  $T_1$ , it follows that

$$(1) \quad 9y^2 - 3\lambda y + 3y = 10$$

(by setting the distance of the line passing through  $MN$  to the line  $L_1$  be equal to 1).

Similarly, since  $MN$  is tangent to  $T_2$ , it follows that

$$(2) \quad 36x^2 + 39x\lambda + 9\lambda^2 = 50.$$

Let  $(x_\lambda, y_\lambda)$  be the positive solution of the equations (1) and (2), for every  $0 \leq \lambda \leq 1$ . It follows easily that  $x_\lambda \geq x_1$  and  $y_\lambda \geq y_0$  for all  $0 \leq \lambda \leq 1$ .

Moreover, the boundary of  $C_4$  is the union of the following three segments:  $\{(x, y_0) \mid x_0 \leq x \leq 3 - y_0\}$ ,  $\{(x_1, y) \mid y_1 \leq y \leq 3 - x_1\}$  and  $\{(x, y) \mid x \geq x_1, y \geq y_0 \text{ and } x + y = 3\}$ , together with the arc  $\alpha$ , represented parametrically and implicitly by (1) and (2), with  $0 \leq \lambda \leq 1$ .

To show that  $C_4$  is convex, it is obviously enough to prove that  $\alpha$  is a convex curve, i.e., that  $d^2y/dx^2 > 0$  at every point of  $\alpha$ .

This is a straightforward computation, which we outline: (1) implies (3) and (4), while (2) implies (5) and (6), where:

$$(3) \quad \frac{dx}{d\lambda} = - \frac{13x + 6\lambda}{24x + 13\lambda},$$

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<sup>1</sup> Suppose there is a candle at each point of  $B_2$ , and every ray passing through the interior of  $T_1$  and  $T_2$  is absorbed;  $C_4$  is then the illuminated part of  $B_1$ .

$$(4) \quad \frac{d^2x}{d\lambda^2} = \frac{25x - 25\lambda(dx/d\lambda)}{(24x + 13\lambda)^2},$$

$$(5) \quad \frac{dy}{d\lambda} = \frac{y}{6y - \lambda + 1},$$

$$(6) \quad \frac{d^2y}{d\lambda^2} = \frac{y + (1 - \lambda)dy/d\lambda}{(6y - \lambda + 1)^2}.$$

Since  $x > 0$ ,  $y > 0$  and  $0 \leq \lambda \leq 1$ , it follows from (3) that  $dx/d\lambda < 0$ , and from (5) that  $dy/d\lambda > 0$ ; these, in turn, imply that  $d^2x/d\lambda^2 > 0$  and  $d^2y/d\lambda^2 > 0$ .

Using the well-known formula

$$\frac{d^2y}{dx^2} = \left( \frac{dx}{d\lambda} \cdot \frac{d^2y}{d\lambda^2} - \frac{dy}{d\lambda} \cdot \frac{d^2x}{d\lambda^2} \right) / \left( \frac{dx}{d\lambda} \right)^3$$

it follows immediately that  $d^2y/dx^2 > 0$  on  $\alpha$ . Moreover,  $d^2y/dx^2 > 0$  implies that  $\alpha$  is not a segment.

As a result,  $C_4$  is a closed convex set which is not a polytope, and the proof of Lemma 1 is completed.

**Proof of Lemma 2.** Let  $D_1, \dots, D_4$  be polytopes, with  $D_i \subseteq C_i$  for all  $1 \leq i \leq 4$ .

$D_4$  is a convex (planar) polygon in  $C_4$ , and  $C_4$  is, by Lemma 1, a closed convex (planar) set which is not a polygon. Therefore, there exists a point  $R$  on the arc  $\alpha$ , such that  $R \notin D_4$ . Since  $D_4$  is convex, there exists the point  $R^*$ , nearest to  $R$  in  $D_4$ . The point  $S = \frac{1}{2}(R + R^*)$  is an interior point of  $C_4$  and  $S \notin D_4$ .

Let  $S'$  be a point in the upper base  $B_2$  of  $P$ , such that the segment  $SS'$  is of distance  $\geq 1$  from the two lines  $L_1$  and  $L_2$ . Since  $S$  belongs to the interior of  $C_4$ , we may assume, without loss of truth, that  $S'$  belongs to the interior of  $B_2$ .

Let  $S_1S'_1$  be a segment, obtained from  $SS'$  by an  $\epsilon$ -push towards the face  $C_3$  of  $P$ , such that  $S_1S'_1 \parallel SS'$ ,  $S_1$  is an interior point of  $C_4 - D_4$ ,  $S'_1$  is an interior point of  $B_2$  and  $S_1S'_1$  is of distance  $> 1$  from both  $L_1$  and  $L_2$ .

The line  $E$ , containing the segment  $S_1S'_1$ , is such that  $E \cap P \neq \emptyset$  and  $E \cap (\bigcup_{i=1}^4 D_i) = \emptyset$ .

This completes the proof of Lemma 2.

**REMARK 1.** The original example, as presented to the Society, involved  $k = 10$  (with  $d = 3$  and  $t = 1$ ); a remark by (a student of) A. J. Hoffman reduced it to  $k = 6$ ; that remark enabled us to further reduce the value of  $k$  to 4.

REMARK 2. By taking the Cartesian product of our example with the  $q$ -dimensional cube  $I^q$ , we derive additional counterexamples in the cases where  $d = q + 3$  and  $t = q + 1$ , for all  $q \geq 1$ .

#### REFERENCES

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