EXTENDING FREE CIRCLE ACTIONS ON SPHERES TO $S^3$ ACTIONS

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Abstract. Let $X$ be a PL homotopy $CP^{k+1}$ corresponding by Sullivan's classification to the element $(N_1, \alpha_1, N_2, \ldots, \alpha_k, N_k)$ of $Z \oplus Z_2 \oplus Z \oplus \cdots \oplus Z_2 \oplus Z$.

Theorem 1. The topological circle action on $S^{4k+2}$ with orbit space $X$ is the restriction of an $S^3$ action with a triangulable orbit space iff $\alpha_i = 0$, $i = 2, \ldots, k$; and $N_1 \equiv 0 \mod 2$; and $\sum (-1)^i N_i = 0$.

If $X$ admits a smooth structure and satisfies the hypotheses of Theorem 1, a certain smoothing obstruction arising from the integrality theorems vanishes for the corresponding $S^3$ action.

In this note we establish some necessary conditions for extending smooth free circle actions on homotopy $(4k+3)$-spheres to free $S^3$ actions ($S^3$ is the group of unit quaternions). It is known that the orbit space of a free circle action on a homotopy sphere $\Sigma^{4k+3}$ is a manifold $X^{4k+2}$ with the homotopy type of complex projective space $CP^{2k+1}$, while the orbit space of an $S^3$ action is a homotopy quaternionic projective space $Y^{2k}$. If the circle action is the restriction of the $S^3$ action (by the maximal torus theorem, the restriction is unique), then $X^{4k+2}$ is an $S^2$ bundle over $Y^{2k}$; in fact this bundle is induced by a homotopy equivalence $Y^{2k} \to QP^k$ from the natural fibering $CP^{2k+1} \to QP^k$. In terms of D. Sullivan's classification up to PL isomorphism of PL homotopy complex projective spaces [7] we are able to state in Theorem 1 necessary and sufficient conditions for a homotopy complex projective space $X^{4k+2}$ to fiber in such a way over some PL homotopy quaternionic projective space. We also show that if $k > 1$ the PL isomorphism class of the orbit space $Y^{2k}$ of an $S^3$ action on $\Sigma^{4k+3}$ is determined by the restricted circle action (the case $k = 1$ is of course equivalent to the four dimensional Poincaré conjecture; in this case Theorem 1 implies that only the standard circle action on $S^7$ extends to an $S^3$ action).

A smooth circle action satisfying the hypotheses of Theorem 1 will thus extend to a topological $S^3$ action with a triangulable orbit space.
It is not known whether additional hypotheses will be necessary to ensure the existence of a smooth extension; as a partial result we define a smoothing obstruction which, although it fails to vanish for many piecewise linear homotopy quaternionic projective spaces, does vanish for orbit spaces of $S^3$ actions which extend smooth circle actions. In particular, if a smooth circle action on a homotopy $S^{11}$ satisfies the hypotheses of Theorem 1, it is shown that the topological action to which it extends can be smoothed.

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1. Fibering homotopy complex projective spaces. Let $E_0$ be the canonical 4-disc bundle over $Q^{P_k}$. The circle group acts on $E_0$ in a fiber preserving manner; let $\overline{E}_0$ denote the orbit space. It is clear that $\overline{E}_0$ is a 3-disc bundle over $Q^{P_k}$ with boundary $\partial \overline{E}_0 = CP^{2k+1}$. Recall that if $M$ is a closed smooth manifold, an $h$-smoothing of $M$ is a homotopy equivalence $f: X \to M$ where $X$ is also a closed smooth manifold. We will say that an $h$-smoothing $f: X \to CP^{2k+1}$ fibers if there is an $h$-smoothing $\tilde{g}: Y \to Q^{P_k}$ such that, denoting $g^*E_0$ by $\overline{E}_Y$, the induced $h$-smoothing $\tilde{g}: \partial \overline{E}_Y \to \partial \overline{E}_0 = CP^{2k+1}$ is concordant to $f$ (i.e. there is a diffeomorphism $c: X \to \overline{E}_Y$ such that $\tilde{g}c \simeq f$). Obviously a free circle action on a homotopy $S^{4k+3}$ extends to an $S^3$ action iff its orbit space fibers.

Fibering is also a sensible notion for an $h$-triangulation (the PL analogue of an $h$-smoothing) of $CP^{2k+1}$. Given an $h$-triangulation $g: Y \to Q^{P_k}$, let $K_Y$ be a triangulation (in the sense of [6]) of $E_Y$. Then an $h$-triangulation $f: X \to CP^{2k+1}$ fibers if it is concordant to an $h$-triangulation $g: \partial K_Y \to \partial \overline{E}_0$ for some $h$-triangulation $g: Y \to Q^{P_k}$. An $h$-triangulation of $CP^{2k+1}$ fibers when and only when the corresponding topological action of the circle on $S^{4k+3}$ extends to an $S^3$ action with a triangulable orbit space.

D. Sullivan [7] has classified $h$-triangulations of $CP^n$ and $Q^{P_n}$ up to concordance in the following way. If $f: X \to CP^n$ is an $h$-triangulation, alter $f$ if necessary to make it transverse regular on $CP^i \subset CP^n$, and let

$$K_i = \text{Ker} f_*: H_i(f^{-1}(CP^i); \mathbb{Z}) \to H_i(CP^i; \mathbb{Z}).$$

If $i$ is even, the intersection form on $K_i$ is symmetric with index $\tau(K_i)$; put $N_{i/2}(f) = \frac{1}{2} \tau(K_i)$. If $i$ is odd, the intersection form is antisymmetric; we denote its Arf invariant by $\alpha_{(i+1)/2}(f)$. Sullivan proved that if $n > 2$ the integers $N_i(f)$ and the mod 2 integers $\alpha_k(f)$ are a complete set of invariants for the concordance class of $f$, and that any
pair \((\alpha_2, \cdots, \alpha_p)\); \((N_1, \cdots, N_q)\), with \(p = \lfloor n/2 \rfloor\) and \(q = \lfloor (n-1)/2 \rfloor\), occurs as the invariants of some \(h\)-triangulation of \(CP^n\). The case of \(QP^m\) is a little simpler. If \(g: Y \to QP^m\) is an \(h\)-triangulation which is transverse regular on \(QP^m\), put
\[
K_i = \ker g_*: H_{2i}(g^{-1}(QP^0)) \to H_{2i}(QP^0), \quad \text{and} \quad M_i(g) = \frac{1}{i} \text{index } (K_i).
\]
The integers \(M_i(g)\) are a complete set of invariants for the concordance class of \(g\) if \(m > 1\). Since \(QP^1 = S^4\) is a spin manifold, \(M_1(g)\) must be even; but any \((m-1)\)-tuple \((M_1, \cdots, M_{m-1})\) with \(M_1\) even will occur as invariants of some \(h\)-triangulation. To simplify our notation, put \(M_0(g) = M_m(g) = 0\).

**Theorem 1.** An \(h\)-triangulation \(f: X \to CP^{2k+1}\) fibers iff the following conditions are satisfied:

1. \(\alpha_i(f) = 0, i = 2, \cdots, k\).
2. \(N_i(f)\) is even.
3. \(\sum_{i=1}^{k} (-1)^i N_i(f) = 0\).

**Proof.** We establish first the necessity of condition 1. Thus suppose \(g: Y \to QP^k\) is an \(h\)-triangulation, and identify \(X\) with \(dK_Y\) via a concordance. In \(E_0\), \(CP^{2i-1}\) bounds the restricted bundle \(E_0| QP^i-1\). It then follows from \([9]\) that the surgery obstruction to making \(f\) \(h\)-regular on \(CP^{2i-1}\) vanishes. This surgery obstruction is just \(\alpha_i(f)\).

The proof is completed by the following

**Lemma.** If \(g: Y \to QP^k\) is an \(h\)-triangulation, and \(\bar{g}: \partial K_Y \to \partial E_0 = CP^{2k+1}\) is the induced \(h\)-triangulation, then
\[
N_i(\bar{g}) = M_i(g) + M_{i-1}(g).
\]

**Proof.** For a PL manifold \(M\), let \(P(M) = 1 + p_1(M) + \cdots\) be the total rational Pontrjagin class. Then, if we put \(X = \partial K_Y\) and \(l: X \to K_Y\) is the inclusion, \(\pi: K_Y \to Y\) the projection, we have \(P(X) = l^*\pi^* [P(Y)g_*P(\bar{E}_0)]\).

Hirzebruch has shown that \(P(\bar{E}_0) = 1 + 4\rho\), where \(\rho\) is the generator of \(H^4(QP^k; \mathbb{Z})\) dual to \(QP^{k-1}\) \([3]\). Put \(y = g^*\rho \otimes 1 \in H^4(Y; Q)\). If \(i\) is the generator of \(H^2(CP^{2k+1}; \mathbb{Z})\) dual to \(CP^{2k}\), put \(x = \bar{g}^*i \otimes 1 \in H^2(X; \mathbb{Z})\), and notice that \(l^*\pi^*(y) = x^2\). Therefore,

\[
(1) \quad P(X) = l^*\pi^* P(Y) \cdot (1 + 4x^2).
\]

Now suppose \(\bar{g}\) is transverse regular on \(CP^{2i} \subset CP^{2k+1}\) and let \(V_i = f^{-1}(CP^{2i})\); notice that index \(V_i = 8N_i(\bar{g}) + 1\).

According to \([8]\) the virtual index formula of Hirzebruch \([4, \S 9]\) is
valid for PL manifolds. Let \( L(M) = 1 + L_1(M) + \cdots \) be the total Hirzebruch class of the manifold \( M \). An application of the virtual index formula to \( X \) yields

\[
\text{index } V_i = 8N_i(\delta) + 1 = \langle (\tanh x)^2(k_i-1)+1L(X),[X] \rangle.
\]

From (1) it follows that

\[
L(X) = l^*\pi^*L(Y) \frac{2x}{\tanh(2x)};
\]

applying a hyperbolic identity,

\[
(tanh x)L(X) = l^*\pi^*L(Y)(1 + \tanh^2 x) \cdot x.
\]

Applying (2),

\[
8N_i(\delta) + 1 = \langle l^*\pi^*L(Y)(\tanh^2 x)^{k-i}(1 + \tanh^2 x), [X] \rangle
\]

The Thom class of the bundle \( E_Y \) is \( \delta x \), where \( \delta : H^2(X; Q) \rightarrow H^3(K_Y, X; Q) \) is the coboundary.

If \([K_Y] \in H_{4n+3}(K_Y, X; Z)\) is an orientation, we may assume \( \partial [K_Y] = [X] \) and \( \partial x \cap [K_Y] = [Y] \). From (3) it follows that

\[
8N_i(\delta) + 1 = \langle \delta \{ l^*\pi^*L(Y)(\tanh^2 x)^{k-i}(1 + \tanh^2 x) \cdot x \}, [K_Y] \rangle
\]

The normal bundle of \( QP^j \subset QP^k \) is \((k-j)E_0\); its Pontrjagin class is \((1+\rho)^{2k-2j}\). If \( g \) is transverse regular to \( QP^j \) and \( W_j = g^{-1}(QP^j) \) then \( h^*P(Y) = P(W_j) \cdot h^*g^*(1+\rho \otimes 1)^{2k-2j}; \ h : W_j \rightarrow Y \) being the inclusion. Thus

\[
h^*L(Y) = L(W_j) \cdot h^* \left( \frac{\sqrt{y}}{\tanh \sqrt{y}} \right)^{2k-2j}
\]

and

\[
\text{index } W_j = \langle L(W_j), [W_j] \rangle
\]

\[
= \langle L(Y) \cdot \left( \frac{\tanh \sqrt{y}}{\sqrt{y}} \right)^{2k-2}, [Y] \rangle
\]

Comparing (4),

\[
8N_i(\delta) + 1 = \text{index } W_i + \text{index } W_{i-1}.
\]

Since index \( W_j = 8M_j(g) \) if \( j \) is odd and \( 8M_j(g) + 1 \) if \( j \) is even, \( N_i(\delta) = M_i(g) + M_{i-1}(g) \).
Remark. The lemma shows that the concordance class of $g$ is determined by that of $g$ if $k > 1$. In terms of $S^3$ actions, the equivariant homeomorphism class of an $S^3$ action on $S^{2k+1}$ ($k > 1$) is determined by the circle action to which it restricts. In case $k = 1$, Theorem 1 may be applied to the classification of smooth circle actions on $S^7$ to show that only the standard circle action extends to an $S^3$ action; see [5].

2. An obstruction to smoothing an $h$-triangulation. Consider the case of an $h$-triangulation $f: M \to S$ of a smooth manifold $S$. Such an $h$-triangulation is concordant to an $h$-smoothing if there is an $h$-smoothing $f': M' \to S$ and a piecewise differentiable homeomorphism $k: M \to M'$ such that $f'k \simeq f$.

$\Omega^\text{spin}_*$ is to be the spin bordism ring. If $X$ is a space, $\Omega^\text{spin}_*(X)$ will denote by analogy with [1] the spin bordism group of $X$. Returning to the $h$-triangulation $f: M \to S$, let $\eta: P \to S$ represent an element of $\Omega^\text{spin}_*(S)$. If we replace both $M$ and $S$ by their products with a sphere of high enough dimension, we can assume $\eta$ is an embedding. Approximate $f$ by a map transverse regular to $\eta(P)$, and put $Q = f^{-1}(\eta(P))$. Then define $\mu_f(\eta) = \langle \hat{A}(Q), [Q] \rangle$, where $\hat{A} = 1 + \hat{A}_1 + \cdots$ is the multiplicative sequence associated with $\frac{1}{2} \sqrt{2}/\sinh \frac{1}{2} \sqrt{2}$.

Proposition. $\mu_f(\eta)$ depends only on the element of $\Omega^\text{spin}_*(S)$ represented by $\eta$ and the homotopy class of $f$. If we allow $\Omega^\text{spin}_*$ to act on $Q$ via $\hat{A}$ then $\mu_f: \Omega^\text{spin}_*(S) \to Q$ is an $\Omega^\text{spin}_*$-homomorphism. If $f$ is concordant to an $h$-smoothing, $\mu_f$ takes integral values.

Proof. The first part follows from a straightforward application of relative transversality and oriented bordism invariance of Pontryagin numbers; details are omitted. If an embedding $\eta: P \to S$ represents an element of $\Omega^\text{spin}_*(S)$ and $\{ U \} \in \Omega^\text{spin}_{m+1}$ then by embedding $U$ in $S'$, $l$ large, $\{ U \} \cdot \{ \eta \}$ is represented by an embedding $U \times P \to S' \times S$; if $f$ is transverse regular to $P$ then id $\times f: S' \times M \to S' \times S$ is transverse regular to $U \times P$, with $(id \times f)^{-1}(U \times P) = U \times Q$. The second part follows then from the multiplicative property of the $\hat{A}$ genus. Finally, if $f$ is concordant to an $h$-smoothing then $Q$ may be taken to be a smooth spin manifold. The statement about the integrality of $\mu_f$ is then a consequence of the differentiable Riemann-Roch theorem; see [4, p. 197].

Theorem 2. If $g: Y \to QP^k$ is an $h$-triangulation such that $\partial Y = \partial \Sigma_{k+1} CP^{2k+1}$ is concordant to an $h$-smoothing, then $\mu_g: \Omega^\text{spin}_*(QP^k) \to Q$ takes integral values.
Proof. It will suffice to check this on $\Omega^{\text{spin}}_4(QP^k)$ since $\mu$ vanishes on components of dimension $\equiv 0 \mod 4$. Let $\eta: P \to QP^k$ represent an element of $\Omega^{\text{spin}}_4(QP^k)$; assume that $\eta$ is an embedding. If $p: CP^{2k+1} \to QP^k$ is the natural projection, let $R = p^{-1}(P)$ and notice that $R$ is a spin manifold of dimension $4i+2$. Approximate $g$ by a map transverse regular on $P$ and let $Q = g^{-1}(P)$, $N = g^{-1}(R)$, and $K_0 = \pi^{-1}(Q)$, so that $\partial K_0 = N$. Since $N$ admits the structure of a smooth spin manifold, $\langle \exp(c)A(N), [N]\rangle$ is integral if $c$ is any element of $H_2(N; \mathbb{Z})$ [4, p. 197]. Since dimension $(N) \equiv 2 \mod 4$, $\langle \exp(c)A(N), [N]\rangle = \langle \sinh(c)A(N), [N]\rangle$.

Now, $P(N) = l^*\pi^*P(Q)(1 + 4x^2)$ where $l$, $\pi$, and $x$ have been restricted to $N$, $K_0$ and $N$, respectively; compare (1). Therefore $l^*\pi^*A(Q) = A(N) \cdot (\sinh x)/x$, and since $\mu_0(\eta) = \langle A(Q), [Q]\rangle$, we have

$$\mu_0(\eta) = \langle \pi^*A(Q), [K_0] \cap \delta x \rangle = \langle \delta(l^*\pi^*A(Q) \cdot x), [K_0]\rangle$$

$$= \langle A(N) \cdot \sinh x, [N]\rangle \in \mathbb{Z}.$$  

In case $\eta$ is not an embedding, this proof may be modified by replacing each of $X$, $Y$, $CP^{2k+1}$, and $QP^k$ by its product with $S^l$ and approximating $\eta$ by an embedding of $P$ into $QP^k \times S^l$.

Remarks. If $I: QP^k \to QP^k$ is the identity map and the $(4k-1)$-skeleton of $Y$ is smooth, then $\mu_0(I)$ is just the Eells-Kuiper invariant [2] of the boundary of a smoothing of $Y_0 = Y - \langle \text{open ball} \rangle$. In particular it follows that an $h$-triangulation of $QP^2$ is concordant to an $h$-smoothing iff $\mu_0(I)$ is integral. For an $h$-triangulation of $QP^3$ the necessary and sufficient condition is that $\mu_0(QP^3)$ be integral and $\mu_0(I)$ be even. For higher dimensions more complicated smoothing obstructions must be considered; generally $\mu_0 \mod 2$ is an obstruction on components of dimension $\equiv 4 \mod 8$ but this refinement may still fail to detect certain nonsmoothable $h$-triangulations of $QP^4$.

Finally, we have not solved the problem of extending smooth circle actions on homotopy spheres to smooth $S^8$ actions for homotopy spheres of dimension greater than seven. In dimension 11, although a topological extension of a smooth circle action can be smoothed if its orbit space is triangulable, it is conceivable that the smoothing would be inconsistent with the circle action.

References

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