A CHARACTERIZATION OF HEREDITARILY INDECOMPOSABLE CONTINUA

ALBERT L. CRAWFORD AND JOHN JOBE

Abstract. In this paper a characterization of a hereditarily indecomposable continuum is stated and proved. The motivation for this characterization is a theorem in a recent article by John Jobe.

In this paper a characterization of a hereditarily indecomposable continuum is stated and proved. The motivation for this characterization is a theorem proved by Jobe in [1]. This result is as follows:

Theorem 1. If $M$ is the 2-finished sum of compact continua, $M_1$ and $M_2$, such that $M_1$ is hereditarily indecomposable and $M_1 \cap M_2 \neq \emptyset$, then there exists at least one point in $M_1 \cap M_2$ which is a limit point of both $(M_1 - M_2)$ and $(M_2 - M_1)$.

Definition. The set $M$ is the 2-finished sum of continua $M_1$ and $M_2$ if $M = M_1 \cup M_2$ and $M_1 - M_2 \neq \emptyset$ and $M_2 - M_1 \neq \emptyset$.

We shall consider the space $S$ to be a Moore space satisfying Axiom 0 and Axiom 1 of R. L. Moore.

First, we suspected that the hypothesis in Theorem 1 was too

Received by the editors February 19, 1970.

AMS 1969 subject classifications. Primary 5438; Secondary 5420.

Key words and phrases. Hereditarily indecomposable continuum, pseudo-arc, Brouwer continuum.

Copyright © 1971, American Mathematical Society

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

205
strong. That is, we suspected that $M_1$ need only be an indecomposable continuum rather than hereditarily indecomposable. The following example in the plane emphasizes the importance of the hypothesis of Theorem 1 as stated. It shows the existence of a compact indecomposable continuum $M_1$ and a compact continuum $M_2$ satisfying the hypothesis of Theorem 1 with the exception that $M_1$ is not hereditarily indecomposable and the conclusion of Theorem 1 is not true.

**Example 1.** Let $M_1$ be the indecomposable continuum in the plane consisting of those semicircles lying above the $x$-axis centered at $(1/2, 0)$ with endpoints in the Cantor ternary set on the $x$-axis together with those semicircles lying below the $x$-axis with centers at $(1/(6 - 3^i), 0)$ and endpoints in the Cantor set. See [3].

Let $M_2$ be the continuum consisting of $M_1$ intersected with the closed upper-half plane together with the line segment from $(1/2, 1/6)$ to $(1/2, 1/2)$.

Then $M_1 - M_2$ lies below the $x$-axis and $M_2 - M_1$ lies above the line $y = 1/6$. Hence there exists no point that is a limit point of both $M_1 - M_2$ and $M_2 - M_1$.

The characterization of a hereditarily indecomposable continuum is in terms of the following defined Property $Q$. The definition of Property $Q$ is motivated by the condition in Theorem 1.

**Definition.** Let $S$ be a Moore space and $M$ a continuum in $S$. Then $M$ has Property $Q$ in $S$ if and only if for every compact continuum $N$ in $S$ such that $N \cap M \neq \emptyset$ and $N \cup M$ is the 2-finished sum of $N$ and $M$, then there exists a point $p \in M \setminus N$ such that $p$ is a limit point of both $M - N$ and $N - M$. A compact continuum $M$ in $S$ has Property $Q$ hereditarily in $S$ if and only if each subcontinuum of $M$ has Property $Q$ in $S$.

**Theorem 2.** Let $T$ be a Moore space and $M$ a compact continuum in $T$. Then $M$ is hereditarily indecomposable if and only if for every function $f$ and Moore space $S$ such that $f$ imbeds $M$ in $S$, then $f(M)$ has Property $Q$ hereditarily in $S$.

**Proof.** Assume the condition of the theorem. Let $M$ be a compact continuum in a Moore space $S$ and suppose that $M$ is not hereditarily indecomposable. Then there exists a decomposable subcontinuum $M' = H \cup K \subseteq M$ where $H$ and $K$ are proper subcontinua of $M'$ and $h \in H \setminus K$. Note that since $S$ is a Moore space then $S \times S$ is also a Moore space. Define maps $f$ and $g$ from $M$ to $S \times S$ as $f(m) = (m, h)$ and $g(m) = (h, m)$ for each $m \in M$. Then, since both $f$ and $g$ imbed $M$ in $S \times S$, $f(M') = M' \times \{h\} = M_1$ and $g(M') = \{h\} \times M' = M_2$ are homeomorphic to $M'$. Let $H_1 = f(H)$ and $K_1 = f(K)$. Then $M_1 = H_1 \cup K_1 \subseteq f(M)$ is decomposable with $(h, h) \in H_1 - K_1$ since $h \in H \setminus K$.
Also the definitions of \( f \) and \( g \) imply that \( M_1 \cap M_2 = \{ (h, h) \} = H_1 \cap M_2 \) since \( H_1 \subset M' \). Thus \( H_1 \cup M_2 \) is a continuum in \( S \times S \). Furthermore, \( (H_1 \cup M_2) - M_1 = M_2 - H_1 \), and hence

\[
(H_1 \cup M_2) - M_1 \subset M_2.
\]

Also \( M_1 - (H_1 \cup M_2) = M_1 - H_1 \subset K_1 \) and

\[
M_1 - (H_1 \cup M_2) \subset K_1.
\]

Since \( K_1 \) and \( M_2 \) are disjoint closed sets, no point is a limit point of both \( M_1 - (H_1 \cup M_2) \) and \( (H_1 \cup M_2) - M_1 \). Hence, \( M_1 = f(M') \) does not have Property Q in \( S \times S \). Therefore, \( f(M) \) does not have Property Q hereditarily in \( S \times S \) which is a contradiction. Therefore, \( M \) is hereditarily indecomposable.

Conversely assume that \( M \) is a compact hereditarily indecomposable continuum in a Moore space \( T \). Let \( f \) be any function that imbeds \( M \) in a Moore space \( S \) and consider \( f(M) \). Since each subcontinuum of \( f(M) \) is itself hereditarily indecomposable, applying Theorem 1 we see that each subcontinuum of \( f(M) \) has Property Q in \( S \). Hence \( f(M) \) has Property Q hereditarily in \( S \) and the condition of the theorem follows.

We thought that in Theorem 2 the condition "for every function \( f \) and Moore space \( S \) such that \( f \) imbeds \( M \) in \( S \), \( f(M) \) has Property Q hereditarily in \( S \)" could be replaced by the condition "\( M \) has Property Q hereditarily in \( T \)." To see that this cannot be done the following example exhibits a Moore space \( T \) and a decomposable compact continuum \( M \) in \( T \) such that \( M \) has Property Q hereditarily in \( T \). Thus, this example complements the statement of Theorem 2.

**Example 2.** Let \( S_1 \) and \( S_2 \) be two pseudo-arcs in the plane constructed from \( (-1, 0) \) to \( (0, 0) \) and \( (0, 0) \) to \( (1, 0) \) respectively such that \( S_1 \cap S_2 = \{ (0, 0) \} \). Let \( p = (0, 0) \). Let \( T \) be the subspace of the plane such that \( T = S_1 \cup S_2 \). Let \( H \) and \( K \) be nondegenerate proper subcontinua of \( S_1 \) and \( S_2 \) respectively such that \( H \cap K = \{ p \} \). Then \( M = H \cup K \) is a decomposable compact continuum that has Property Q hereditarily in \( T \).

Suppose \( N \) is a subcontinuum of \( T \) such that \( N \cap M \neq \emptyset \) and \( N \cup M \) is the 2-finished sum of \( N \) and \( M \). Clearly \( p \in N \) and we may assume that \( (N \cap S_1) \subset H \) and \( K \subset (N \cap S_2) \). Then \( M - N = H - (N \cap S_1) \) and \( N - M = (N \cap S_2) - K \). Because \( H \) and \( N \cap S_2 \) are pseudo-arcs, \( p \) is a limit point of both \( H - (N \cap S_1) \) and \( (N \cap S_2) - K \). Since \( p \in M \cap N \) it has been verified that \( M \) has Property Q in \( T \).

Now if \( M' \) is a subcontinuum of \( M \), then either (a) \( M' \subset S_1 \) or (b) \( M' \subset S_2 \) or (c) \( M' - S_1 \neq \emptyset \) and \( M' - S_2 \neq \emptyset \). In cases (a) and (b), \( M' \)}
is hereditarily indecomposable and hence has Property Q in T. In case (c), $M'$ has Property Q in T by the method used to show that $M$ has Property Q in T. Therefore, $M$ has Property Q hereditarily in T.

Since $M$ is decomposable the example is verified.

There is a natural question suggested by Theorem 2 and Example 2—namely, is the property in question really a property of the embedding or a property of the containing space? More precisely, if $M$ is a compact continuum, $S$ is a Moore space, and $f$ and $g$ are two embeddings of $M$ in $S$, does $f(M)$ have Property Q hereditarily in $S$ if and only if $g(M)$ has Property Q hereditarily in $S$? Example 3 gives a negative answer to this question.

**Example 3.** Let $T$ and $M$ be as defined in Example 2. By Theorem 2 there exists a Moore space $T_1$ and an imbedding function $f$ from $M$ to $T_1$ such that $f(M)$ does not have Property Q hereditarily in $T_1$. The space $T_1$ can be picked such that $T \cap T_1 = \emptyset$. Let $g$ be the imbedding from $M$ to $T$ such that $g(m) = m$, $m \in M$. Example 2 reveals that $g(M)$ has Property Q hereditarily in $S$.

Let $S = T \cup T_1$ and $A$ be open in $S$ if and only if $A$ is the union of an open set in $T$ and an open set in $T_1$. Note that $S$ is a Moore space. It follows that $f$ and $g$ can be thought of as imbeddings of $M$ into $S$ and clearly $g(M)$ has Property Q hereditarily in $S$ while $f(M)$ does not have Property Q hereditarily in $S$.

The authors have been able to find two other characterizations of hereditarily indecomposable continua. These two can be found in [2] and [4].

**Bibliography**


**Oklahoma State University, Stillwater, Oklahoma 74074**