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EXPANSIONS OF VECTORS IN A BANACH SPACE RELATED TO GAUSSIAN MEASURES

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ABSTRACT. We study norm convergent expansions of vectors in a Banach space B related to a Gaussian measure on B, and as a corollary obtain the convergence in supremum norm of the partial sums of the Karhunen-Loève expansion of a Gaussian process having continuous sample paths.

1. **Introduction.** In [2] L. Gross introduced the concept of an abstract Wiener space and in [4] additional results were obtained. The purpose of this note is to point out that the expansions for vectors in a Banach space obtained in [4] converge in a much stronger sense than is indicated there. As a corollary it will follow that if $\{x_t : 0 \leqq t \leqq T\}$ is a Gaussian process with continuous sample paths, then the partial sums of the Karhunen-Loève expansion converge in the supremum norm with probability one.

The main result is Theorem 3.1 and the application to Gaussian processes is in Theorem 4.1. §2 contains necessary preparations. The proofs of Lemmas 2.1 and 2.3 occur in [5], and the terminology concerning integration over infinite-dimensional linear spaces is that used in [2].

2. **Notation and basic preliminaries.** Throughout the paper B will denote a real separable Banach space with norm $\|\cdot\|_B$. B^* denotes the topological dual of B and $\mathcal{A}_B$ the Borel subsets of B. Our first objective is to define a particular inner product on B. This is the result of the next lemma. Its proof appears in [5].

LEMMA 2.1. *If B is a real separable Banach space with norm $\|\cdot\|_B$, then there exists an inner product $(\ ,\ )$ on B such that the norm generated by $(\ ,\ )$ is weaker than $\|\cdot\|_B$. Further, if $\tilde{H}$ is the completion of B under the inner product norm, then $\mathcal{A}_B \subseteq \mathcal{A}_{\tilde{H}}$ where $\mathcal{A}_{\tilde{H}}$ denotes the Borel subsets of $\tilde{H}$.*

A measure μ on the Borel subsets of a real locally convex linear space $\mathcal{L}$ is defined to be *Gaussian* if, for every continuous linear func-

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tional T on $\mathfrak{L}$, $T(x)$ has a Gaussian distribution with mean zero. We will always assume that μ is not concentrated on some finite-dimensional subspace of $\mathfrak{L}$ since it is the only case of interest here.

The next lemma follows immediately.

LEMMA 2.2. *If μ is a Gaussian measure on $(B, \mathfrak{A}_B)$ and $(\tilde{H}, \mathfrak{A}_{\tilde{H}})$ is defined as in Lemma 2.1, then the equation $\mu(A) = \mu(A \cap B)$, $A \in \mathfrak{A}_{\tilde{H}}$, extends μ to a Gaussian measure on $(\tilde{H}, \mathfrak{A}_{\tilde{H}})$.*

Now if μ is a Gaussian measure on B and hence on $\tilde{H}$ it follows that there exists a nonnegative symmetric trace class operator A on $\tilde{H}$ (such an operator is often called an S-operator) such that

$$(Ax, x) = \int_{\tilde{H}} (x, y)^2 d\mu(y)$$

for $x \in \tilde{H}$, and that μ is uniquely determined on $(\tilde{H}, \mathfrak{A}_{\tilde{H}})$ by the operator A. These results are well known and appear, for example, in [6]. Further, if A is an S-operator on $\tilde{H}$ it is known that

$$A(\cdot) = \sum_k \lambda_k(\cdot, \phi_k)\phi_k \tag{2.1}$$

where $\{\phi_k\}$ is an orthonormal sequence in $\tilde{H}$, and $\lambda_k > 0$, $\sum_k \lambda_k < \infty$.

Consequently, given A we are able to define the Hilbert space

$$H = \left\{ x \in \tilde{H} : x \in \operatorname{span}\{\phi_1, \phi_2, \cdots\}, \sum_k \frac{(x, \phi_k)^2}{\lambda_k} < \infty \right\} \tag{2.2}$$

where for $x, y \in H$ the inner product is

$$(x, y)_H = \sum_k \frac{(x, \phi_k)(y, \phi_k)}{\lambda_k} \tag{2.3}$$

and the inner product $(\ ,\ )$ stands for the product on $\tilde{H}$.

The next lemma is also proved in [5] and hence again is only stated here.

LEMMA 2.3. *If μ is a Gaussian measure on B which induces the S-operator A on $\tilde{H}$ of the form* (2.1) *and H is defined as in* (2.2), *then*

(1) $H \subseteq B \subseteq \tilde{H}$;

(2) *the measure μ is the restriction to $(B, \mathfrak{A}_B)$ of the Gaussian measure on $\tilde{H}$ induced by the canonical normal distribution on H.*

In fact, using the proof of Lemma 2.3 in [5], it is easy to see that if E denotes the closure of H in $\tilde{H}$ with respect to the norm $\|\cdot\|_{\tilde{H}}$, then $\mu(E) = 1$ (here μ is being viewed as a measure on $\tilde{H}$). Consequently, if

$\mathcal{A}_E$ denotes the Borel subsets of E, then $(E, \mathcal{A}_E, \mu)$ is an abstract Wiener space with generating Hilbert space H in the sense indicated in [2] and [4]. That is, E is a real separable Hilbert space (of infinite dimension since μ is assumed not to be concentrated on any finite-dimensional subspace of B), μ is a Gaussian measure on $\mathcal{A}_E$ induced by the canonical normal distribution on H, $\|\cdot\|_{\hat{H}}$ is a measurable norm on it, and E is the completion of H under $\|\cdot\|_{\hat{H}}$.

In [4] it is shown that if $f \in H$ then we can define a "stochastic inner product" $(x, f)^{\sim}$ which exists with μ-measure one on E. Further, $(x, f)^{\sim}$ can be taken to be Borel measurable and it has a Gaussian distribution with mean zero and variance $\|f\|_H^2$. Since $\mu(E \cap B) = 1$ it follows that $(x, f)^{\sim}$ exists with μ-measure one on B. Thus if $\{\alpha_k\}$ is any orthonormal sequence in H it follows that $\{(x, \alpha_k)^{\sim}\}$ are independent Gaussian random variables on $(B, \mathcal{A}_B, \mu)$ each with mean zero and variance one.

LEMMA 2.4. *Suppose μ is a Gaussian measure on $(B, \mathcal{A}_B)$ and H is defined as in Lemma 2.3. Further, assume $T \in B^*$, $\{\alpha_k\}$ is any complete orthonormal subset of H, and*

$$x_N = \sum_{k=1}^{N} (x, \alpha_k)^{\sim} \alpha_k \qquad (N = 1, 2, \cdots).$$

Then

$$\lim_N T(x_N) = T(x) \tag{1}$$

with μ-measure one on B and

$$\mu(\overline{H}) = 1 \tag{2}$$

where $\overline{H}$ is the closure of H in B.

PROOF. Let E be defined as above. Since $T \in B^*$ it follows that T is Borel measurable and linear when restricted to $E \cap B$. Further, T can then be extended to be linear on E by the usual transfinite argument, and since $\mu(E - (E \cap B)) = 0$ it follows that any extension of T from $E \cap B$ to E is measurable on E with respect to the completed sigma-algebra obtained from μ and $\mathcal{A}_E$. Thus by Theorem 3.2 of [4] we have $\lim_N T(x_N) = T(x)$ with μ-measure one on E. Since $\mu(B \cap E) = 1$ conclusion (1) now follows.

To see (2) notice that $\overline{H} \in \mathcal{A}_B \subseteq \mathcal{A}_{\hat{H}}$ and hence $\overline{H} \cap E \in \mathcal{A}_E$. Now the remark following Theorem 3.1 of [4] assures us that $\mu(\overline{H} \cap E) = 0$ or 1.

Assuming $\mu(\overline{H} \cap E) = 0$ implies $\mu(\overline{H}) = 0$ and thus we can find (by the Hahn-Banach Theorem) a $T \in B^*$ such that $T \equiv 0$ on $\overline{H}$ and T is

positive on some open sphere V in $B-\overline{H}$. Choose V so that $\mu(V)>0$ (this is possible since $B-\overline{H}$ is open, $\mu(B-\overline{H})=1$, and B is separable). Then extend T as above to be linear on E. Now by Lemma 3.2 of [4] it follows that T restricted to H is a bounded linear functional and its H-norm is given by

$$\int_E [T(x)]^2 d\mu(x) = \int_B [T(x)]^2 d\mu(x) > 0.$$

This is a contradiction since $T\equiv 0$ on H thus $\mu(\overline{H})=1$.

3. We now use the results of §2 and an important result due to Itô and Nisio [3] to prove our main theorem. When we say a Gaussian measure μ on B has a generating Hilbert space H we mean that B, μ, H are related as in §2 and, in particular, as in Lemma 2.3.

THEOREM 3.1. *Suppose μ is a Gaussian measure on B. Then μ has a generating Hilbert space $H\subseteq B$ such that*

(1) $\mu(\overline{H})=1$ *($\overline{H}$ is the closure of H in B);*

(2) *for any complete orthonormal sequence $\{\alpha_n\}$ in H and expansions*

$$x_N = \sum_{k=1}^{N} (x, \alpha_k)^{\sim}\alpha_k \qquad (N = 1, 2, \cdots)$$

we have

$$\lim_N \|x - x_N\|_B = 0 \tag{3.1}$$

with μ-measure one on B.

PROOF. That μ has a generating Hilbert space H such that $H\subseteq B$ and $\mu(\overline{H})=1$ follows from Lemma 2.4. In addition, if $T\in B^*$ Lemma 2.4 tells us that $\lim_N T(x_N)=T(x)$ with μ-measure one. This implies, using Theorem 4.1–e of [3], that (3.1) holds with μ-measure one and the proof is complete.

4. **Application to Gaussian processes.** Let $\{x_t: 0\leqq t\leqq T\}$ be a separable Gaussian process with covariance function $R(s, t)$ and mean zero. We will restrict our attention to mean-continuous processes which is equivalent to assuming that $R(s, t)$ is continuous on $[0, T]\times[0, T]$, and we can and do assume that $\{x_t\}$ has its sample paths in $\mathcal{L}_2[0, T]$.

Since $R(s, t)$ is continuous and nonnegative definite it has the eigenfunction expansion $\sum_k \lambda_k\phi_k(s)\phi_k(t)$ which converges uniformly on $[0, T]\times[0, T]$, the eigenvalues λ_k are positive numbers such that $\sum_k \lambda_k<\infty$, and the eigenfunctions $\{\phi_k(t)\}$ are continuous orthonor-

mal elements of $\mathcal{L}_2[0, T]$. Furthermore, $\{x_t\}$ induces a Gaussian measure μ on the Hilbert space $\mathcal{L}_2[0, T]$ which is uniquely determined by the S-operator

$$Ax(t) = \int_0^T R(s, t)x(s)ds.$$

The paths of the process have the Karhunen-Loève expansion

$$x(t) = \sum_{k=1}^{\infty} \lambda_k^{1/2}(x, \Gamma_k)\phi_k(t) \tag{4.1}$$

where $\lambda_k^{1/2}\Gamma_k = \phi_k$ and the inner product appearing in (4.1) is the usual one for $\mathcal{L}_2[0, T]$. Then, as is well known, the series in (4.1) converges with μ-measure one to $x(t)$ when the norm is the usual one on $\mathcal{L}_2[0, T]$.

However, many Gaussian processes have continuous sample paths and then there is a Gaussian measure μ on $C[0, T]$ which is uniquely determined by $R(s, t)$ (recall we assumed the mean is zero). Since the functions $\phi_n(t)$ $(n = 1, 2, \cdots)$ are continuous it is natural to ask if (4.1) converges uniformly with μ-measure one. Indeed, this is the case in special situations (see, for example, [3]) and our next theorem indicates that it always happens.

Now assume μ is a Gaussian measure on the Banach space $C = C[0, T]$ determined by the covariance function $R(s, t)$ and let $\|\cdot\|_C$ denote the sup-norm on C. We define the Hilbert space H as all functions which are in the span of the eigenfunctions $\{\phi_k(t)\}$ of $R(s, t)$ and such that

$$\sum_k \frac{(x, \phi_k)^2}{\lambda_k} < \infty \tag{4.2}$$

where $(x, y) = \int_0^T x(s)y(s)ds$. The inner product on H is then given by (2.3) and it is clear that $H \subseteq C$. The functions $\alpha_k = \lambda_k^{1/2}\phi_k$ form a complete orthonormal sequence in H and since $(x, \alpha_k)_H = (x, \phi_k)/\lambda_k^{1/2}$ we can write (4.1) in the form

$$x(t) = \sum_{k=1}^{\infty} (x, \alpha_k)_H \alpha_k(t). \tag{4.3}$$

Letting G denote the closure of H in C with respect to the norm $\|\cdot\|_C$ it follows from [1] that $\mu(G) = 1$. Now G is a closed subspace of C and hence μ is a Gaussian measure on G.

Thus if $L(x)$ is a continuous linear functional on G it is easy to see that L restricted to H is continuous as a functional on H, and, arguing

as in [4, Theorem 3.1], it follows that $L(x)=(x,f)^{\sim}$ with μ-measure one where $f\in H$ and the random variable $(x,f)^{\sim}$ is defined by the series

$$\sum_{j=1}^{\infty}(x,\alpha_j)_H(f,\alpha_j)_H. \tag{4.4}$$

Hence $(x,f)^{\sim}$ is Gaussian with mean zero and variance $\|f\|_H^2$. Then, as in Lemma 3.2 of [4], we see

$$L(x)=(x,f)_H=\sum_k(x,\Gamma_k)_H(f,\Gamma_k)_H$$

for x in H and $\{\Gamma_k(t)\}$ any complete orthonormal set in H. Now if $J(x)=\sum_{k=1}^{\infty}(x,\Gamma_k)^{\sim}(f,\Gamma_k)_H$ it follows that $J(x)=(x,f)^{\sim}=L(x)$ with μ-measure one. Thus the partial sums

$$x_N(t)=\sum_{k=1}^{N}(x,\Gamma_k)^{\sim}\Gamma_k(t)\qquad (N=1,2,\cdots) \tag{4.5}$$

are such that

$$\begin{aligned}\lim_N L(x_N)&=\lim_N\sum_{k=1}^{N}(x,\Gamma_k)^{\sim}L(\Gamma_k)\\&=\lim_N\sum_{k=1}^{N}(x,\Gamma_k)^{\sim}(\Gamma_k,f)_H=L(x)\end{aligned}$$

with μ-measure one. The next theorem now follows directly from the Itô-Nisio result used in the proof of Theorem 3.1.

THEOREM 4.1. *Suppose $\{x_t:0\leqq t\leqq T\}$ is a Gaussian process with continuous sample paths and that μ is the measure induced on C. Let H be defined as in* (4.2) *and suppose $\{\Gamma_k(t)\}$ is any complete orthonormal sequence in H. Then the partial sums of* (4.5) *converge uniformly to $x(t)$ with μ-measure one. In particular, if $\Gamma_k=\lambda_k^{1/2}\phi_k$ we see the Karhunen-Loève expansion for $x(t)$ converges in the uniform norm with μ-measure one.*

ADDED IN PROOF. The recent work of L. Shepp and H. Landau and also of X. Fernique showing that the norm has certain exponential moments of second order with respect to any Gaussian measure on a real separable Banach space enables one to use the elegant technique of J. B. Walsh (Proc. Amer. Math. Soc. **18** (1967), 129–132) to obtain the results of this paper. In addition, R. M. Dudley has kindly pointed out to the author that Theorem 4.6 of his paper in J. Functional Analysis **1** (1967), 290–330 has a relationship to the results of

this paper. Finally, a recent paper (Proc. Amer. Math. Soc. **25** (1970), 890–895) by G. Kallianpur and N. Jain also deals with these ideas.

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