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GENERAL EMBEDDING PROPERTIES OF ABSOLUTE BOREL AND SOUSLIN SPACES

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ABSTRACT. In a recent paper, S. Willard established several characterizations of absolute metric G_α-spaces in terms of the Borel character they possessed as subspaces of certain compact Hausdorff spaces; and he asks whether a similar result holds for the F_α-spaces. In the present paper, we show that for a metric space X the following are equivalent for $\alpha \geqq 2$: (1) X is an absolute metric F_α-space, (2) X is a $Z_\alpha \cap G_\delta$-set (i.e., a *Baire* set of class α intersected with a G_δ-set) in some compactification, (3) X is an $F_\alpha \cap G_\delta$-set in every completely regular Hausdorff embedding, (4) X is an absolute F_α-space with respect to the class of all perfectly normal spaces. These properties remain equivalent when "F_α" and "Z_α" are replaced by "Souslin." Necessary and sufficient conditions for a metric space to be an F_α-set in all its compactifications are found and, throughout, extensions to spaces which are not necessarily metrisable are provided.

1. **Introduction.** If $\mathcal{E}$ is a family of subsets of a fixed set X and α is a countable ordinal, we define the family $\mathcal{E}_\alpha$ as follows: $\mathcal{E}_0 = \mathcal{E}$ and, assuming the families have been defined for all ordinals $<\alpha$, we define $\mathcal{E}_\alpha$ to be the family of all sets of the form $\bigcup_{n=1}^{\infty} A_n$ or $\bigcap_{n=1}^{\infty} A_n$ where each A_n belongs to $\mathcal{E}_{\alpha_n}$ for some $\alpha_n < \alpha$. When X is a topological space and $\mathcal{E}$ is the family of all open sets in X, respectively the family of all closed (zero) sets, it is customary to replace $\mathcal{E}_\alpha$ by $G_\alpha(X)$, respectively $F_\alpha(X)$ $(Z_\alpha(X))$, or simply G_α, or F_α etc. if the space X is understood. If X is a metric space, then $\bigcup G_\alpha(X) = \bigcup F_\alpha(X) = \bigcup Z_\alpha(X)$, where the union is over all countable ordinals α, and the sets belonging to this collection are called the *Borel sets* of X. We note that this classification of the Borel sets—into classes which form an increasing transfinite sequence of type ω_1—differs slightly from the usual classification given for metric spaces (see e.g. [6, p. 345]).[1] However, it is easily verified that the results established in this paper are equally valid with respect to the usual classification. A member of the family $\mathcal{E}_\alpha$ will be called an $\mathcal{E}_\alpha$-*set in* X.

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[1] For a given metric space, if we let F'_α and G'_α denote the families of sets of class α relative to the traditional classification scheme, then one easily sees that $F_\alpha = F'_\alpha \neq G'_\alpha = G_\alpha$ for finite α, whereas $F_\alpha = F'_\alpha \cup G'_\alpha = G_\alpha$ for $\omega_0 \leqq \alpha < \omega_1$.

A topological space X is said to be an *absolute metric G_α-set* (abbreviated AG_α(metric)) provided X is metrisable and is a G_α-set in every metric space in which it is topologically embedded. Spaces which are AF_α(metric) are defined analogously. And, of course, the definitions can be applied, in an obvious way, to more general classes of spaces.

With this terminology, a classical result due in part to Alexandrov, Čech, and Sierpiński, states that the following properties of a metric space X are equivalent: (1) X is completely metrisable, (2) X is a $G_1 = G_\delta$-set in some complete metric space, (3) X is a G_δ-set in its Stone-Čech compactification βX, and (4) X is an AG_δ (metric). Classical too is the theorem of M. Lavrentiev [**6**, p. 431] which states that properties (2) and (4) above remain equivalent when G_δ is replaced by G_α (or F_α), for $\alpha \geqq 1$ (respectively $\alpha \geqq 2$). More recently, S. Willard in [**9**] generalized the equivalence of (3) and (4) as follows:

THEOREM (WILLARD). *For a metrisable space X, the following are equivalent for $\alpha \geqq 1$:*

(1) *X is an AG_α (metric),*

(2) *X is a G_α-set in βX,*

(3) *X is a G_α-set in some compactification of X.*

Later in [**10**] *the theorem was extended to include:*

(4) *X is a G_α-set in every compactification of X, and*

(5) *X is a G_α-set in its closure in every completely regular Hausdorff embedding.*

The question was also raised: "Can a result similar to the theorem above be obtained for F_α-sets?" (see [**9**, p. 324]). The purpose of this note is to provide the following analogue of Willard's theorem for F_α-sets:

MAIN THEOREM. *For a metrisable space X, the following conditions are equivalent for $\alpha \geqq 2$:*

(1) *X is an AF_α(metric),*

(2) *X is a $Z_\alpha \cap G_\delta$-set (i.e., a Z_α-set intersected with a G_δ-set) in some compactification,*

(3) *X is a $Z_\alpha \cap G_\delta$-set in βX,*

(4) *X is an $F_\alpha \cap G_\delta$-set in every compactification,*

(5) *X is an $F_\alpha \cap G_\delta$-set in every completely regular Hausdorff embedding,*

(6) *X is an AF_α (perfectly normal spaces).*[2]

[2] The referee has brought to my attention the paper by Z. Frolík [**3**] in which the equivalence of (1), (2), and (3) is stated (Theorem 3(1), p. 782) in a less precise form (without any reference to classes). The same theorem covers the equivalence of (1), (2), and (3) of our corollary to Theorem 3 in §5.

If, in addition, we assume in (1) that X is separable, then the conditions remain equivalent with "$Z_\alpha \cap G_\delta$" and "$F_\alpha \cap G_\delta$" replaced by "Z_α" and "F_α" respectively (corollary to Theorem 2, §4). Both theorems are first proven for a certain class of perfectly normal spaces which properly contains the class of all metrisable spaces (Theorems 1 and 2, §§3 and 4). Finally, analogues for absolute Souslin spaces are given (Theorems 3 and 4, §5).

2. **Preliminaries.** All spaces considered are assumed to be completely regular and Hausdorff (abbreviated CR-T_2) unless we state the contrary.

Let X be a fixed space and suppose that to each finite sequence $(n_1, \cdots, n_p)$ of natural numbers there corresponds a closed subset $F(n_1, \cdots, n_p)$ of X. Then the set

$$S = \bigcup_{(n_p)} \bigcap_{p=1}^{\infty} F(n_1, \cdots, n_p),$$

where the union is over all sequences $(n_p) = (n_1, n_2, \cdots)$ of natural numbers, is called a *Souslin set in X* [5, p. 203]. By an *absolute* (*metric*) *Souslin set* we mean a (metric) space which is a Souslin set in every (metric) space in which it can be embedded. We now list several well-known properties of Borel and Souslin sets.

2.1 For all $E \subset X$, if E is a Z_α-set in X, then E is an F_α-set in X, and this in turn implies that E is a Souslin set in X.

2.2 If $E \subset X \subset Y$, then E is an F_α-set (G_α-set, Souslin set) in X if and only if it is of the form $X \cap M$ where M has the corresponding property in Y.

2.3 The complement of a G_α-set is an F_α-set, and every F_α-set (respectively Z_α-set) is an F_β-set (Z_β-set) for all $\beta \geqq \alpha$.

2.4 A finite intersection or union of F_α-sets (respectively Z_α-sets, G_α-sets) is again of that class.

2.5 If $f: X \to Y$ is continuous, and $E \subset Y$ is an F_α-set (Z_α-set, G_α-set, Souslin set) in Y, then $f^{-1}(E)$ has the same property with respect to X.

2.6 (Choquet [1]) Every Souslin set in a compact space is Lindelöf.

We will also have need of the following basic lemma.

2.7 [4, p. 92] If f is a continuous mapping of a space E into a space Y, whose restriction to a dense subspace X is a homeomorphism, then $f(E-X) \subset Y - f(x)$.

3. **Proof of Main Theorem.** Let us first recall that a topological space X is said to be *complete in the sense of Čech* if X is CR-T_2 and satisfies one of the following three equivalent conditions [2, p. 142]:

(i) X is a G_δ-set in some compactification,
(ii) X is a G_δ-set in βX,
(iii) X is a G_δ-set in every compactification.

Also, a space X is called *perfectly normal* if it is normal and if every closed subset is a G_δ-set; or equivalently, if every closed subset is a zero set (see [2, Problem 1.G, p. 60]). We single out those spaces X which are densely embeddable in perfectly normal complete spaces in the sense of Čech. Every metrisable space has this property, since every metric space is perfectly normal [2, p. 175] and each is densely embeddable in some complete metric space, which is also complete in the sense of Čech (see §1).[3] We will prove that for such spaces X our Main Theorem is valid. More precisely, we prove the following:

THEOREM 1. *Let X be densely embedded in a space X^* which is perfectly normal and complete in the sense of Čech.*

Then the following conditions are equivalent for $\alpha \geqq 2$:
(1) *X is an F_α-set in X^*,*
(2) *X is a $Z_\alpha \cap G_\delta$-set in some compactification,*
(3) *X is a $Z_\alpha \cap G_\delta$-set in βX,*
(4) *X is an $F_\alpha \cap G_\delta$-set in every compactification,*
(5) *X is an $F_\alpha \cap G_\delta$-set in every* CR-T_2 *space in which it is embedded,*
(6) *X is an AF_α (perfectly normal spaces).*

PROOF. (1)→(2). We show by transfinite induction that for every F_α-set F in X^* there exists a Z_α-set Z in βX^* such that $F = Z \cap X^*$. For $\alpha = 0$, F is a closed (and hence zero) set in X^*, say $F = f^{-1}(\{0\})$, where f is a continuous bounded real-valued function on X^*. Let f_β be the Stone extension of f to βX^* and set $Z = f_\beta^{-1}(\{0\})$. Then, clearly, $F = Z \cap X^*$, and so Z is the desired zero set. Assuming the above is true for all ordinals $<\alpha$ (α a countable ordinal >0), let A_n be an F_{α_n}-set in X^*, $\alpha_n < \alpha$, $n = 1, 2, \cdots$. Accordingly, let Z_n be a Z_{α_n}-set in βX^* such that $A_n = Z_n \cap X^*$. Then, if $F = \bigcup_{n=1}^\infty A_n$, $Z = \bigcup_{n=1}^\infty Z_n$ is a Z_α-set in βX^* and $F = Z \cap X^*$. Similarly, if $F = \bigcap_{n=1}^\infty A_n$, we may take $Z = \bigcap_{n=1}^\infty Z_n$, and the induction is complete. It follows that if X is an F_α-set in X^*, then $X = Z \cap X^*$ for some Z_α-set Z in βX^*. Since X^* is a G_δ-set in βX^* and βX^* is a compactification of X, we have shown that (2) follows from (1).

[3] Not every space with this property is metrisable. The following example of a compact, perfectly normal but nonmetrisable space was suggested by the referee: Topologise the lexicographically ordered square with the order topology, and consider the closed, and hence compact, subspace formed by the top and bottom edges. One then shows, with some difficulty, that this subspace is perfectly normal but not second countable (and hence nonmetrisable).

(2)→(3). Let $X = Z \cap G$ where Z is a Z_α-set and G is a G_δ-set in some fixed (but arbitrary) compactification bX of X. Let $f: \beta X \to bX$ be the Stone extension of the embedding $X \to bX$. By 2.5, $f^{-1}(Z \cap G) = f^{-1}(Z) \cap f^{-1}(G)$ is a $Z_\alpha \cap G_\delta$-set in βX; and, by 2.7, $f^{-1}(Z \cap G) = X$. Hence (3). The more difficult proof that (3)→(4) requires a lemma.

LEMMA 1. *Let X be a* CR-T_2 *space and let bX be a compactification of X. Let $f: \beta X \to bX$ be the Stone extension of the embedding $X \to bX$ and let τ denote the topology on the set βX consisting of all sets of the form $f^{-1}(W)$ where W is an open subset of bX (note that $(\beta X, \tau)$ is compact but not necessarily* CR-T_2).

(a) *If U is open in βX and $V = \mathrm{Int}_\tau(U)$, then $U \cap X = V \cap X$.*

(b) *If Z is a zero set in βX, then $\mathrm{Cl}_\tau(Z \cap X) \cap X = Z \cap X$, and there exists a G_δ-set G in $(\beta X, \tau)$ such that $X \subset G$ and $\mathrm{Cl}_\tau(Z \cap X) \cap G \subset Z$.*

(c) *A necessary and sufficient condition for $E \subset \beta X$ to be an F_α-set (respectively G_α-set, Souslin set) in $(\beta X, \tau)$, is that it be of the form $f^{-1}(M)$ where M has the corresponding property in bX.*

PROOF. To prove (a), it is enough to show that $U \cap X \subset V \cap X$, so let $x \in U \cap X$. Since $f(\beta X - X) \subset bX - X$ by 2.7, the set $f(\beta X - U) = f(\beta X - X) \cup f(X - U)$ does not contain $f(x)$. Thus $f(x) \in bX - f(\beta X - U)$, from which it follows that

$$x \in f^{-1}(bX - f(\beta X - U)) = \beta X - f^{-1}(f(\beta X - U)) \subset U.$$

Since $\beta X - U$ is compact, $f(\beta X - U)$ is compact, and so $bX - f(\beta X - U)$ is open in bX. Thus we have shown that $x \in \mathrm{Int}_\tau(U) \cap X = V \cap X$.

To prove (b), note first that zero sets are G_δ-sets [2, p. 15], so $Z = \bigcap_{n=1}^\infty U_n$ where each U_n is open in βX. Let $V_n = \mathrm{Int}_\tau U_n$ and observe that, by (a) above, $Z \cap X = (\bigcap_{n=1}^\infty V_n) \cap X$. Now put $G = (\beta X - \mathrm{Cl}_\tau(Z \cap X)) \cup (\bigcap_{n=1}^\infty V_n)$. It follows from 2.4 that G is a G_δ-set with respect to $(\beta X, \tau)$. Since the τ subspace topology on X coincides with its original topology, we have $\mathrm{Cl}_\tau(Z \cap X) \cap X = Z \cap X$, and thus $\beta X - \mathrm{Cl}_\tau(Z \cap X) \supset X - (Z \cap X)$. This, together with the fact that $\bigcap_{n=1}^\infty V_n \supset Z \cap X$, proves that $G \supset X$. Finally, we have $\mathrm{Cl}_\tau(Z \cap X) \cap G = \mathrm{Cl}_\tau(Z \cap X) \cap (\bigcap_{n=1}^\infty V_n) \subset Z$, as follows easily from the definition of G and the fact that $\bigcap_{n=1}^\infty V_n \subset Z$. This completes the proof of (b).

That the condition in (c) is sufficient follows immediately from property 2.5. On the other hand, the necessity of the condition follows easily from the fact that a set E is τ-closed (resp. τ-open) in βX if and only if it is of the form $f^{-1}(M)$ where M is closed (resp. open) in bX, and the fact that the inverse image operator preserves intersections and unions. The routine induction arguments are omitted. This completes the proof of Lemma 1.

We now turn to the proof that (3)$\rightarrow$(4). Let X be a space, and let Z be a Z_α-set and H and G_δ-set, both with respect to βX, such that $X=Z\cap H$. Let bX, f, and τ have the same meaning attached to them in Lemma 1. We will show first that X is an $F_\alpha\cap G_\delta$-set with respect to $(\beta X, \tau)$ (cf. the strategy used in [9]). Since H is a G_δ-set, $H=\bigcap_{n=1}^\infty H_n$ where each H_n is open in βX. Let $W_n=\mathrm{Int}_\tau H_n$, $n=1, 2, \cdots$. Then, from (a) of Lemma 1, $X=Z\cap W$ where $W=\bigcap_{n=1}^\infty W_n$.

We now assert that given any Z_α-set Z in βX there exists an F_α-set F and a G_δ-set G, both with respect to $(\beta X, \tau)$, such that $X\subset G$, $F\cap G\subset Z$, and $F\cap X=Z\cap X$. For $\alpha=0$, Z is a zero set in βX, and so the existence of sets F and G is an immediate consequence of Lemma 1 part (b). Assuming the proposition has been shown for all ordinals $<\alpha$ (α a countable ordinal >0), let $Z=\bigcup_{n=1}^\infty A_n$ where A_n is a Z_{α_n}-set, $\alpha_n<\alpha$, $n=1, 2, \cdots$. By the induction hypothesis, there exists F_{α_n}-sets F_n and G_δ-sets G_n, with respect to $(\beta X, \tau)$, such that $X\subset G_n$, $F_n\cap G_n\subset A_n$, and $F_n\cap X=A_n\cap X$, for each n. Now set $F=\bigcup_{n=1}^\infty F_n$ and $G=\bigcap_{n=1}^\infty G_n$. Then F is an F_α-set and G is a G_δ-set in $(\beta X, \tau)$, $X\subset G$, $F\cap G\subset\bigcup_{n=1}^\infty F_n\cap G_n\subset\bigcup_{n=1}^\infty A_n=Z$, and $F\cap X=\bigcup_{n=1}^\infty F_n\cap X=\bigcup_{n=1}^\infty A_n\cap X=Z\cap X$. Similarly, if $Z=\bigcup_{n=1}^\infty A_n$, we may take $F=\bigcap_{n=1}^\infty F_n$ and $G=\bigcap_{n=1}^\infty G_n$, and so the assertion is proved.

Returning to the equation $X=Z\cap W$, let F and G satisfy the above assertion for Z. Then $X=Z\cap X=F\cap X\subset F$, and so we have $X\subset F\cap G\cap W$. On the other hand, $(F\cap G)\cap W\subset Z\cap W=X$. Since F is an F_α-set and $G\cap W$ is a G_δ-set in $(\beta X, \tau)$, we have shown that $X=F\cap(G\cap W)$ is an $F_\alpha\cap G_\delta$-set in $(\beta X, \tau)$. Now, by (c) of Lemma 1, there exists an F_α-set F' and a G_δ-set H' in bX such that $F=f^{-1}(F')$ and $G\cap W=f^{-1}(H')$. Consequently, $X=f^{-1}(F'\cap H')$. But, by property 2.7, $X=f(X)$, and so we have $X=F'\cap H'$, proving that X is an $F_\alpha\cap G_\delta$-set in bX. This completes the proof of the implication (3)$\rightarrow$(4).

(4)$\rightarrow$(5). Assume X satisfies (4) and let X be a subspace of the (CR-T_2) space Y. Then $\beta(\mathrm{Cl}_Y X)$ is a compactification of X, so X is an $F_\alpha\cap G_\delta$-set in $\beta(\mathrm{Cl}_Y X)$. From 2.2 (§2), we conclude that X has the same property in $\mathrm{Cl}_Y X$. Hence, by another application of 2.2, X is of the form $(F\cap H)\cap\mathrm{Cl}_Y X$ where $F\cap H$ is an $F_\alpha\cap G_\delta$-set in Y. It follows that $X=(F\cap\mathrm{Cl}_Y X)\cap H$ is an $F_\alpha\cap G_\delta$-set in Y. Moreover, if Y is also perfectly normal, then H is an $F_{\sigma\delta}$-set; thus, since $\alpha\geqq 2$, X is an F_α-set in Y. Hence we have shown that (4)$\rightarrow$(5)$\rightarrow$(6).

Since the implication (6)$\rightarrow$(1) is trivial, all the equivalences in Theorem 1 have been established.

Our Main Theorem is, of course, an immediate consequence of Theorem 1; we need only take for X^* any metric completion of X.

REMARK. It would be interesting to know whether or not "$Z_\alpha \cap G_\delta$-set" can be replaced by "$F_\alpha \cap G_\delta$-set" in conditions (2) and (3) of Theorem 1. On the other hand, can one characterize absolute Z_α-sets by replacing F_α by Z_α in statements (1), (4), (5), and (6) of Theorem 1? (Cf. [**7**, Remark 2.2].)

4. It is natural at this juncture to ask "When is a space an F_α-set in all of its compactifications?" The answer takes a surprisingly simple form.

THEOREM 2. *Let X be densely embeddable in a space X^* which is perfectly normal and complete in the sense of Čech. Then the following conditions are equivalent for $\alpha \geqq 2$:*

(1) *X is a Lindelöf space and an AF_α*(CR-T_2),
(2) *X is a Lindelöf space and an F_α-set in X^*,*
(3) *X is a Z_α-set in some compactification,*
(4) *X is a Z_α-set in βX,*
(5) *X is an F_α-set in every compactification,*
(6) *X is an AF_α*(CR-T_2).

REMARK. The equivalence of (3) and (4) is essentially known (see [**7**, Theorem 2.6, p. 692]).

We first prove a basic lemma.

LEMMA 2. *If X is a Lindelöf space and is an $F_\alpha \cap G_\delta$-set (or a $Z_\alpha \cap G_\delta$-set) in some* CR-T_2 *space Y, then X is an F_α-set (Z_α-set) in Y provided $\alpha \geqq 2$.*

PROOF. Suppose $X = F \cap H$ where F is an F_α-set (Z_α-set) and H is a G_δ-set in Y. Let $H = \bigcap_{n=1}^\infty H_n$, where each H_n is open in Y. Since Y is CR-T_2, we can cover X, for each n, by interiors of zero-sets $\{Z_{xn}\}$ in Y such that $Z_{xn} \subset H_n$ for each $x \in X$ [**4**, §3.2]. Since X is Lindelöf, X is contained in the union Z_n of a countable subcollection of the Z_{xn}. It follows that $\bigcap_{n=1}^\infty Z_n$ is a Z_2-set in Y, and that

$$X \subset F \cap \left(\bigcap_{n=1}^{\infty} Z_n \right) \subset F \cap \left(\bigcap_{n=1}^{\infty} H_n \right) = F \cap H = X.$$

Therefore, $X = F \cap (\bigcap_{n=1}^\infty Z_n)$ and hence, since $\alpha \geqq 2$, is an F_α-set (respectively, a Z_α-set) in Y.

To prove Theorem 2 we note first that (1)→(2) since X^* is CR-T_2. Assuming (2), it follows from Theorem 1 that X is a $Z_\alpha \cap G_\delta$-set in some compactification, and hence, by the lemma, that X is a Z_α-set in the same compactification. Hence (2)→(3). If X is a Z_α-set in bX, a compactification of X, then surely X is a $Z_\alpha \cap G_\delta$-set in bX. And so, by Theorem 1, X is a $Z_\alpha \cap G_\delta$-set in βX and hence a Z_α-set in βX, by Lemma 2 (which applies in view of properties 2.1 and 2.6). Thus (3)→(4). Similarly, using Theorem 1 and Lemma 2, we deduce that (4)→(5) and (5)→(6). Again, (6)→(1) by 2.1 and 2.6 of §2. This completes the proof of Theorem 2.

COROLLARY. *For a metrisable space X, the following conditions are equivalent for $\alpha \geqq 2$:*

(1) *X is separable and an AF_α(metric),*

(2) *X is a Z_α-set in some compactification,*

(3) *X is a Z_α-set in βX,*

(4) *X is an F_α-set in every compactification,*

(5) *X is an AF_α*(CR-T_2).

PROOF. Since all metrisable spaces satisfy the hypothesis of Theorem 2, the equivalence of (2) through (5) follows immediately from the theorem. On the other hand, since a metric space is separable if and only if it is Lindelöf, it is clear from Theorem 2 that (5) →(1). Moreover, (1) implies that X is an F_α-set, and hence a Z_α-set, in the compact space $I^{\aleph_0}$ (the Hilbert cube), and so (1)→(2). The proof is thus complete.

REMARK. For an interesting discussion of AF_α(metric) spaces for $\alpha < 2$, the reader is referred to the paper of Stone [8].

5. **Absolute Souslin sets.** We prove here analogues of Theorems 1 and 2 for absolute Souslin sets. We first state as a lemma the following fact proved in [1, p. 78].

LEMMA (CHOQUET). *If $f: X \to Y$ is a continuous mapping between two compact spaces and $S \subset X$ is a Souslin set in X, then $f(S)$ is a Souslin set in Y.*

THEOREM 3. *Let X be densely embeddable in a space X^* which is perfectly normal and complete in the sense of Čech. Then the following conditions are equivalent:*

(1) *X is a Souslin set in X^*,*

(2) *X is a Souslin set intersected with a G_δ-set in some compactification,*

(3) *X is a Souslin set intersected with a G_δ-set in βX,*

(4) *X is a Souslin set intersected with a G_δ-set in every compactification,*

(5) *X is a Souslin set intersected with a G_δ-set in every* CR-T_2 *embedding,*

(6) *X is an absolute Souslin set in the class of all perfectly normal spaces.*

PROOF. (1)→(2). If X is a Souslin set in X^*, then, by 2.2, X is of the form $S\cap X^*$, where S is a Souslin set in βX^*. Since X^* is a G_δ-set in βX^*, the implication is proved.

(2)→(3). Let $X=S\cap G$ where S is a Souslin set and G is a G_δ-set in bX, a compactification of X. If $f:\beta X\rightarrow bX$ denotes the Stone extension of the embedding $X\rightarrow bX$, then, by 2.5, $f^{-1}(S\cap G)=f^{-1}(S)\cap f^{-1}(G)$ is a Souslin set intersected with a G_δ-set in βX. However, by 2.7, $f^{-1}(S\cap G)=X$, proving (3).

(3)→(4). Again, suppose that $X=S\cap G$, where S is a Souslin set and G is a G_δ-set in βX, and let bX be a given compactification of X. Letting f denote again the Stone extension of the embedding $X\rightarrow bX$, there exists, by parts (a) and (c) of Lemma 1, a G_δ-set H in bX such that $X=S\cap f^{-1}(H)$ (cf. the proof of the corresponding part in Theorem 1). Hence $f(X)=f(S)\cap H$. By the preceding lemma, $f(S)$ is a Souslin set in bX, and $X=f(X)$ by 2.7. It follows that X is a Souslin set intersected with a G_δ-set in bX.

The proofs that (4)→(5), (5)→(6), and (6)→(1) are entirely analogous to the proofs given for the corresponding implications in Theorem 1, and so we omit the details. This completes the proof of Theorem 3.

COROLLARY. *For a metrisable space X, conditions* (2)–(6) *of Theorem* 3 *are all equivalent to*:

(1) *X is an absolute metric Souslin set.*[2]

The proof of our final theorem and corollary differs in no essential way from the proofs given already for Theorem 2 and its corollary. The reader should have no trouble in supplying the details.

THEOREM 4. *Let X be densely embeddable in a space X^* which is perfectly normal and complete in the sense of Čech. Then the following conditions are equivalent*:

(1) *X is a Lindelöf space and an absolute Souslin set for the class of all* CR-T_2 *spaces,*

(2) *X is a Lindelöf space and a Souslin set in X^*,*

(3) *X is a Souslin set in some compactification,*

(4) *X is a Souslin set in βX,*

(5) *X is a Souslin set in every compactification,*

(6) *X is an absolute Souslin set for the class of all* CR-T_2 *spaces.*

COROLLARY. *For a metrisable space X, conditions* (3)–(6) *of Theorem* 4 *are all equivalent to*:

(1) *X is separable and an absolute metric Souslin set.*

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