PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, No. 2, February 1971

NOTE ON THE EMBEDDING OF MANIFOLDS IN EUCLIDEAN SPACE

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ABSTRACT. M. Hirsch and independently H. Glover have shown that a closed k-connected smooth n-manifold M embeds in R^{2n-j} if M_0 immerses in R^{2n-j-1}, $j\leqq 2k$ and $2j\leqq n-3$. Here M_0 denotes M minus the interior of a smooth disk. In this note we prove the converse and show also that the isotopy classes of embeddings of M in R^{2n-j} are in one-one correspondence with the regular homotopy classes of immersions of M_0 in R^{2n-j-1}, $j\leqq 2k-1$ and $2j\leqq n-4$.

1. **Introduction.** Let M be a closed k-connected smooth n-manifold and let M_0 denote M minus the interior of a smooth disk. Hirsch [6] and Glover [2] have shown that M embeds in R^{2n-j} if M_0 immerses in R^{2n-j-1}, $j\leqq 2k$ and $2j\leqq n-3$. In this note we will apply the technique of [2] to prove

(1.1) THEOREM. *Suppose $j\leqq 2k$ and $2j\leqq n-3$. Then M embeds in R^{2n-j} if and only if M_0 immerses in R^{2n-j-1}.*

(1.2) THEOREM. *Suppose $j\leqq 2k-1$ and $2j\leqq n-4$. There is a one-one correspondence between the isotopy classes of embeddings of M in R^{2n-j} and the regular homotopy classes of immersions of M_0 in R^{2n-j-1}.*

For $k>0$, these extend results of Haefliger and Hirsch [5] over a range of $(k-1)$-dimensions.

Let ν denote a normal m-plane bundle over M, $m>n$, ν_0 its restriction to M_0 and $\nu_0(j+1)$ the associated bundle with fibre $V_{m,m-n+j+1}$. According to Hirsch (see [7, Theorem 1.2]), the regular homotopy classes of immersions of M_0 in R^{2n-j-1} are in one-one correspondence with the vertical homotopy classes of sections to $\nu_0(j+1)$, which we denote by $C(\nu_0(j+1))$. In §5 we will show that $C(\nu_0(j+1)) \cong [M_0;\ V_{m,m-n+j+1}]$. Thus we have the following classification theorem.

(1.3) COROLLARY. *Suppose $j\leqq 2k-1$ and $2j\leqq n-4$. If M embeds in R^{2n-j}, the isotopy classes of embeddings of M in R^{2n-j} are in one-one correspondence with the elements of $[M_0;\ V_{m,m-n+j+1}]$, $m>n$.*

Received by the editors July 10, 1969.

AMS 1970 subject classifications. Primary 57D40.

Key words and phrases. Tubular neighborhood, deleted product, equivariant map, obstruction theory, Postnikov resolution.

[1] The first author was supported by National Science Foundation Grant GP 24498, the second by Grant GP 5252.

2. **The deleted product.** Let $T(M)$ denote the tangent bundle of M and Δ the diagonal of $M\times M$. The open disk of radius r in R^n will be denoted by D_r^n. Fix $p\in M$ and let $\phi:R^n\to M$ be a diffeomorphism onto an open neighborhood of p such that $\phi(0)=p$. Using a suitable partition of unity, construct a Riemannian metric ρ on $T(M)$ such that $T(\phi):D_1^n\times R^n\to T(M)$ is metric preserving. Now choose ϵ so that $0<\epsilon<1/3$ and $E_\rho:T_\epsilon(M)\to M\times M$ by $E_\rho(v_x)=(\exp_\rho(v_x),\exp_\rho(-v_x))$ embeds $T_\epsilon(M)$ as a tubular neighborhood of the diagonal. Here $T_\epsilon(M)$ denotes the disk bundle of radius ϵ. Let U, V and W denote the image under ϕ of D_ϵ^n, $D_{2\epsilon}^n$ and $D_{3\epsilon}^n$ respectively. We may assume that ϵ is small enough that

(2.1) $T_\epsilon(M)\cap(M\times\overline{U}\cup\overline{U}\times M)\subset\phi(D_1^n)\times\phi(D_1^n)$. Now the following properties are easily established.

(2.2) $T_\epsilon(M-V)\cap(M\times U\cup U\times M)=\Phi$.

(2.3) $M\vee M=M\times\{p\}\cup\{p\}\times M$ *is a strong deformation retract of* $M\times U\cup U\times M\cup \operatorname{Int} T_\epsilon(V)$.

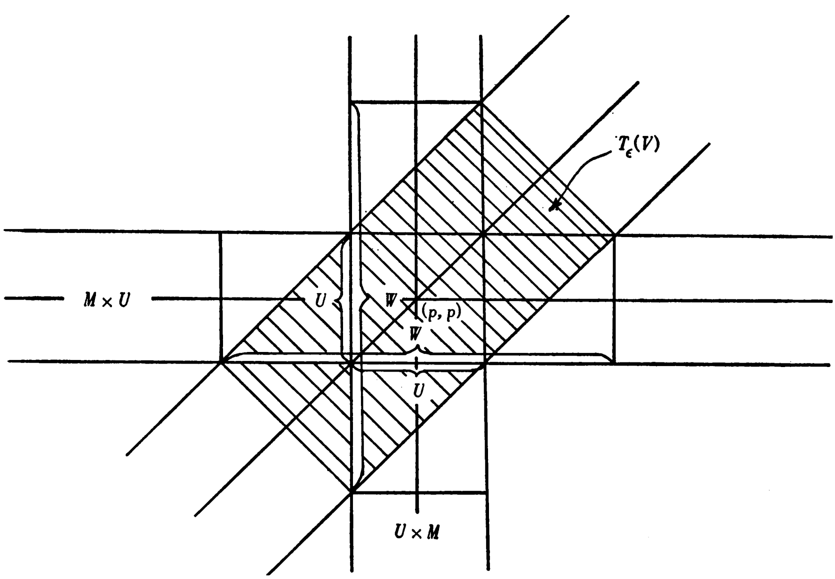
(2.4) $\Delta\cup(\{p\}\times M)$ *is a strong deformation retract of*

$$\operatorname{Int} T_\epsilon(M)\cup U\times M\cup W\times U.$$

(2.5) $\Delta\cup(M\times\{p\})$ *is a strong deformation retract of*

$$\operatorname{Int} T_\epsilon(M)\cup M\times U\cup U\times W.$$

(2.6) $X=M\times M-(\operatorname{Int} T_\epsilon(M)\cup U\times W\cup W\times U)$ *is a strong deformation retract of* $M\times M-\Delta$.



Set $A = X\cap(M\times U\cup U\times M)$ and note that (2.2) implies that $S_\epsilon(M-V)\subset X-A$, where $S_\epsilon(M-V)$ is the sphere bundle of radius ϵ over $M-V$.

(2.7) LEMMA. $H^{2n-j}(X-A, S_\epsilon(M-V))=0, j\leqq 2k+1$.

PROOF. We have

$$\begin{aligned} H^{2n-j}(X - A, S_\epsilon(M - V)) &= H^{2n-j}(X - A \cup T_\epsilon(M - V), T_\epsilon(M - V)) \\ &= H^{2n-j}(X - A \cup T_\epsilon(M - V)). \end{aligned}$$

Now note that

$$X - A \cup T_\epsilon(M - V) = M \times M - (M \times U \cup U \times M \cup \operatorname{Int} T_\epsilon(V)).$$

Then by Poincaré duality and (2.3) we have

$$\begin{aligned} &H^{2n-j}(X - A \cup T_\epsilon(M - V)) \\ &\qquad = H_j(M \times M, M \times U \cup U \times M \cup \operatorname{Int} T_\epsilon(V)) \\ &\qquad = H_j(M \times M, M \vee M) = 0, \qquad j \leqq 2k + 1. \end{aligned}$$

3. **Equivariant maps.** In this section we will record a few facts about equivariant maps which will be needed later. If X and Y are spaces with an involution, let $E(X, Y)$ denote the set of equivariant homotopy classes of equivariant maps from X to Y.

(3.1) *Suppose Y is a finite* CW-*complex with a fixed point free cellular involution α. Let $q\geqq 0$. There is a finite* CW-*complex Z with a fixed point free cellular involution and equivariant inclusion $Y\subset Z$ such that*

(a) *Z is $(q-1)$-connected,*

(b) *$Z-Y$ consists of cells of dimension $\leqq q$.*

PROOF. Suppose Y is $(s-1)$-connected. Let $[f_1]\cdots[f_t]$ generate $\pi_s(Y)$ and let

$$Z_1 = D^{s+1} \cup_{f_1} Y \cup_{\alpha f_1} D^{s+1}.$$

Extend α to an involution on Z_1 by interchanging the cells. Let $i_1: Y\to Z_1$ denote the inclusion. Then Z_1 is $(s-1)$-connected and $\pi_s(Z_1)$ is generated by $i_{1\#}[f_2]\cdots i_{1\#}[f_t]$. Continue in this way.

(3.2) *Let X and Y be finite* CW-*complexes with a fixed point free cellular involution and let $f: X\to Y$ be equivariant. Suppose $f^*: H^q(Y)\to H^q(X)$ is an isomorphism, $q>t$, and is onto $q=t$. Then $f^\#: E(Y, S^q)\to E(X, S^q)$ is one-one, $q>t$, and is onto, $q=t$.*

This is well known from obstruction theory.

4. **Proof of (1.1) and (1.2).** Take Y to be $S_\epsilon(M-V)$ and q to be

$2n-2k-2$ in (3.1) and let $i: S_\epsilon(M-V) \to Z$ denote the inclusion. Because of statement (b) in (3.1) we have

(4.1) LEMMA. *$i^*: H^{2n-j}(Z) \to H^{2n-j}(S_\epsilon(M-V))$ is an isomorphism, $j \leqq 2k+1$, and is onto, $j=2k+2$.*

By (2.7), i can be extended to an equivariant map

$$\tilde{\lambda}: X - A \to Z. \tag{4.2}$$

We will now apply a construction of Glover [2]. Notice that

$$X = (X - A) \cup (U \times (M - W)) \cup ((M - W) \times U).$$

Let $\Sigma(Z)$ denote the suspension of Z with the suspended involution and extend $\tilde{\lambda}$ to an equivariant map

$$\lambda: X \to \Sigma(Z) \tag{4.3}$$

by

$$\lambda(tv, y) = t\tilde{\lambda}(v, y) + (1 - t)s^+$$

and

$$\lambda(y, tv) = t\tilde{\lambda}(y, v) + (1 - t)s^-$$

where $v \in \mathrm{Bdy}(\overline{U})$, $y \in M - W$, $0 \leqq t \leqq 1$, and s^+, s^- are the north and south pole of $\Sigma(Z)$ respectively.

(4.4) LEMMA. *$\lambda^*: H^{2n+1-j}(\Sigma(Z)) \to H^{2n+1-j}(X)$ is an isomorphism, $j \leqq 2k+1$, and is onto, $j = 2k+2$.*

PROOF. Let

$$X^+ = (X - A) \cup (U \times (M - W)),$$
$$X^- = (X - A) \cup ((M - W) \times U),$$

and let $C^+(Z)$ and $C^-(Z)$ denote the upper and lower cone in $\Sigma(Z)$. By (2.5)

$$\begin{aligned} H^{2n-j}(X^+) &\cong H_j(M \times M, \mathrm{Int}\, T_\epsilon(M) \cup M \times U \cup U \times W) \\ &\cong H_j(M \times M, \Delta \cup (\{p\} \times M)) = 0, \qquad j \leqq 2k+1. \end{aligned} \tag{4.5}$$

Similarly, by (2.6)

$$H^{2n-j}(X^-) = 0, \qquad j \leqq 2k+1. \tag{4.6}$$

By (2.7) and (4.1)

$$\lambda^*: H^{2n-j}(Z) \to H^{2n-j}(X - A) \tag{4.7}$$

is an isomorphism, $j \leqq 2k+1$, and is onto, $j=2k+2$. The proof is now completed by comparing the Mayer-Vietoris sequence of $\{C^+(Z); C^-(Z)\}$ with that of $\{X^+; X^-\}$.

Now consider the maps

$$\Sigma(S_\epsilon(M - V)) \xrightarrow{\Sigma(i)} \Sigma(Z) \xleftarrow{\lambda} X \subset M \times M - \Delta. \tag{4.8}$$

Because of (2.6), X and $M \times M - \Delta$ are equivariantly homotopy equivalent. Combining (4.1) and (4.4) with (3.2) and Theorem (2.5) of [1] we have

(4.9) *Suppose* $j \leqq 2k$. *There is an equivariant map* $M \times M - \Delta \to S^{2n-j-1}$ *if and only if there is an equivariant map* $S_\epsilon(M-V) \to S^{2n-j-2}$.

(4.10) *Suppose* $j \leqq 2k$. *There is a one-one correspondence* $E(M \times M - \Delta, S^{2n-j}) \to E(S_\epsilon(M-V), S^{2n-j-1})$.

Theorems (1.1) and (1.2) now follow from the work of Haefliger [3] and Haefliger and Hirsch [4].

5. **Proof of (1.3).** This is proved by applying the following lemma to $\nu_0(j+1)$.

(5.1) Lemma. *Suppose* $\beta = (E, B, p)$ *is a fibration which admits a cross-section. Assume that* B *is* k*-connected and* $(n-k)$*-coconnected and the fibre* F *is* $(n-2k-1)$*-connected. Then* $C(\beta) \cong [B; F]$.

Proof. Set $q = n-k-1$ and let $\beta^{(q)} = (E^{(q)}, B, p^{(q)})$ with fibre $F^{(q)}$ denote the qth term in a Postnikov resolution of β. Let $\delta: B \to E^{(q)}$ be a cross-section and choose base-points $b_0 \in B$ and $x_0 = \delta(b_0) \in F^{(q)}$. The partial lifting $g: B \vee F^{(q)} \to E^{(q)}$ of the projection $\rho_1: B \times F^{(q)} \to B$ defined by $g(b, x_0) = \delta(b)$, $b \in B$, and $g(b_0, x) = x$, $x \in F^{(q)}$, extends to a lifting $\tilde{g}: B \times F^{(q)} \to E^{(q)}$ since $H^r(B \times F^{(q)}, B \vee F^{(q)}; \pi_{r-1}(F^{(q)})) = 0$, $r \geqq 0$. Therefore $\beta^{(q)}$ is weakly fibre homotopy equivalent to the product $(B \times F^{(q)}, B, \rho_1)$ so that $C(\beta^{(q)}) \cong [B; F^{(q)}]$.

Finally, since B is $(q+1)$-coconnected $C(\beta) \cong C(\beta^{(q)})$ and $[B; F] \cong [B; F^{(q)}]$.

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