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A NOTE ON FOURIER MULTIPLIERS

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ABSTRACT. A short proof is given of de Leeuw's restriction result for multipliers.

In this note we prove directly the following result of de Leeuw (Proposition 3.2 in [1]).

THEOREM. *Let $m(x, y)$ be a Fourier multiplier for $L^p(R^{i+j})$. Then for almost every x, $m_x(y) \equiv m(x, y)$ is a Fourier multiplier for $L^p(R^j)$ and the multiplier norm of m_x does not exceed that of m. In particular, the restriction is possible for each x such that (x, y) is a Lebesgue point of m for almost all $y \in R^j$.*

To prove this we recall that the (necessarily) bounded measurable function m is a Fourier multiplier for $L^p(R^n)$ if and only if there is a constant C such that, for $f, g \in C_0^\infty(R^n)$,

$$(*) \qquad \left| \int m(x)\hat{f}(x)\hat{g}(-x)dx \right| \leqq (2\pi)^n C \|f\|_p \|g\|_{p'},$$

where $\hat{f}(x) = \int f(y)e^{-ix\cdot y}dy$ and $1/p + 1/p' = 1$. The best constant C is then the norm of the operator K defined by $m\hat{f} = (Kf)\hat{\ }$, and we write $|m|_p$ for this quantity.

REMARK. The inequality $(*)$ might be taken as the definition of Fourier multiplier instead of: $m(L^p)\hat{\ } \subseteqq (L^p)\hat{\ }$ for $1 \leqq p \leqq 2$, a duality argument for $p > 2$.

If $p = 1$ or $p = 2$ the result is obvious since Fourier transforms of L^1 functions restrict and since the Fourier multipliers for L^2 are the essentially bounded measurable functions.

In the other cases let $f, \varphi \in C_0^\infty(R^i)$, $g, \psi \in C_0^\infty(R^j)$. We assume at first that m is continuous. Apply $(*)$ and Fubini, using $f(x)g(y)$ for $f(x, y)$, $\varphi(x)\psi(y)$ for $g(x, y)$, to deduce that

$$I(x) \equiv \frac{1}{(2\pi)^j} \int m(x, y)\hat{g}(y)\hat{\psi}(y)dy$$

is a Fourier multiplier for $L^p(R^i)$, with $|I|_p \leqq |m|_p \|g\|_p \|\psi\|_{p'}$. Since

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$\|m\|_\infty \leqq |m|_p$ (for $|m|_p = |m|_{p'}$, $\|m\|_\infty = |m|_2 \leqq |m|_p^{1/2}|m|_{p'}^{1/2}$ by the Riesz interpolation theorem), we have

$$|I(x)| \leqq |m|_p \|g\|_p \|\psi\|_{p'},$$

which is $(*)$ for m_x.

To remove the restriction of continuity, form the convolution $a_\epsilon m$ of m with ϵ^{-n} times the characteristic function of the ϵ-cube centered at the origin. A change in the order of integration in $(*)$ gives $|a_\epsilon m|_p \leqq |m|_p$. When (x, y) is a Lebesgue point of m for almost all y in R^j, $(a_\epsilon m)_x \to m_x$ pointwise and boundedly as $\epsilon \to 0$. From $(*)$, the preceding paragraph, and dominated convergence it follows that m_x is a Fourier multiplier for $L^p(R^j)$, and $|m_x|_p \leqq |m|_p$.

Reference

1. K. de Leeuw, *On L^p multipliers*, Ann. of Math (2) **81** (1965), 364–379. MR **30** #5127.

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