PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, No. 2, February 1971

A NOTE ON A PAPER OF PALAIS

L. JONKER

ABSTRACT. If Φ^p denotes the space of smooth alternating p-forms on the C^∞ manifold M, we are interested in finding the spaces of $\boldsymbol{R}$-linear maps from Φ^p to Φ^q that commute with the diffeomorphisms of M. For a compact manifold M these spaces were found by R. S. Palais. In this note we find them for noncompact M.

Let V, W be the spaces of sections of two tensor bundles over a connected C^∞ manifold M of dimension n. In a paper [3] appearing in Trans. Amer. Math. Soc. **92** (1959), R. S. Palais studied a number of the spaces $\mathcal{g}(V, W)$ of the $\boldsymbol{R}$-linear maps $V \rightarrow W$ natural with respect to the action on V and W of the group G of diffeomorphisms on M. We sketch the definition of this action: If $g \in G$ we define $R_g : \Gamma(TM) \rightarrow \Gamma(TM)$ to be the differential of g, and $R_g : \Gamma(T^*M) \rightarrow \Gamma(T^*M)$ to be the dual of the differential of g^{-1}. For the other spaces of tensor fields R_g is defined as the obvious tensor product of these two $\boldsymbol{R}$-linear isomorphisms. A map $c: V \rightarrow W$ is then called natural if $cR_g = R_g c$ for all $g \in G$.

Palais was particularly interested in the case when V and W are the space Φ^p of p-forms and the space Φ^q of q-forms respectively, $0 \leqq p, q \leqq n$. There are three cases when $\mathcal{g}(\Phi^p, \Phi^q)$ is nonempty: $\mathcal{g}(\Phi^p, \Phi^{p+1})$ consists of the exterior derivative d and its real multiples; $\mathcal{g}(\Phi^p, \Phi^p)$ consists of all multiples of the identity; if M is compact, orientable, and irreversible[1] then $\mathcal{g}(\Phi^n, \Phi^0)$ consists of the real multiples of integration of n-forms over M. In fact, $\mathcal{g}(\Phi^p, \Phi^q)$ is empty in all other cases.

For some of his theorems, Palais had to assume that M was compact. The purpose of this note is to lift that restriction. In particular we have the following new theorems:

$$\mathcal{g}(\Phi^0, \Phi^0) = \text{constant multiples of the identity},$$

$$\mathcal{g}(\Phi^p, \Phi^0) = 0 \quad \text{if } 0 < p < n,$$

$$\mathcal{g}(\Phi^n, \Phi^0) = 0 \quad \text{if } M \text{ is noncompact.}$$

Using the method of this note, one can in fact generalize all the

Received by the editors March 27, 1970.

AMS 1969 *subject classifications*. Primary 5345.

Key words and phrases. Differential form, exterior derivative, integration of n-forms, deRham cohomology.

[1] M is called irreversible if there is no orientation reversing diffeomorphism on M.

theorems in [3] to the case of noncompact manifolds. The results of this note are contained in the author's Ph.D. thesis [2] at the University of Toronto.

For $0 \leqq p < n$ let Ψ_1^p be the subset of Φ^p consisting of the following p-forms: $\phi \in \Psi_1^p$ if and only if there is a relatively compact open sphere U in some coordinate system $(x^1, \cdots, x^n)$ on M so that $\operatorname{supp} \phi \subset U$ and so that in a neighbourhood of $m = (0, 0, \cdots, 0)$ we have $\phi = x^{p+1}\, dx^1 \wedge \cdots \wedge dx^p$.

We shall say that ϕ is centered at m. In [3] elements of Ψ_1^p are called basic forms of the first kind. A family $\{\phi_\alpha\}$ of elements in Ψ_1^p will be called *disjoint* if the closures of the corresponding spherical coordinate neighbourhoods U_α are mutually disjoint and the family $\{U_\alpha\}$ is locally finite. Clearly, the family $\{\phi_\alpha\}$ may then be summed. Let Ψ_2^p denote the collection of all p-forms obtained as the sum of a disjoint family of elements in Ψ_1^p. If $\phi \in \Psi_2^p$, $\phi = \sum_\alpha \phi_\alpha$, $\phi_\alpha \in \Psi_1^p$, $\{\phi_\alpha\}$ disjoint, ϕ_α centered at m_α, then we say that ϕ is centered at the family of points $\{m_\alpha\}$. Let Φ_c^p denote the space of p-forms of compact support.

1. LEMMA [3, THEOREM 7.4]. *Every element in Φ_c^p, $0 \leqq p < n$, is a finite sum of elements in Ψ_1^p.*

2. LEMMA. *Ψ_2^p generates Φ^p if $0 \leqq p < n$.*

PROOF. Let $\{U_{\alpha_1}\}_{\alpha_1 \in A_1}, \cdots, \{U_{\alpha_{n+1}}\}_{\alpha_{n+1} \in A_{n+1}}$ be a collection of spherical relatively compact coordinate neighbourhoods the union of whose elements covers M, and which have the further property that for each i, $\{\overline{U}_{\alpha_i}\}_{\alpha_i \in A_i}$ is a disjoint and locally finite collection. The fact that such coverings of M exist is of central importance in this note. A proof may be found in [1]. Let $\bigcup_{i=1}^{n+1} \{f_{\alpha_i}\}_{\alpha_i \in A_i}$ be a subordinate partition of unity, $\operatorname{supp} f_{\alpha_i} \subset U_{\alpha_i}$. Let $w \in \Phi^p$, and let $w_{\alpha_i} = f_{\alpha_i} w$. Then by the proofs of Theorems 7.1, 7.3 in [3] it is seen that

$$w_{\alpha_i} = f_{\alpha_i} w = \sum_{j=1}^{N} \overset{j}{w}_{\alpha_i}, \qquad \overset{j}{w}_{\alpha_i} \in \Psi_1^p \cup \{0\}$$

where 0 denotes the identically zero p-form and where $N = 4\binom{n}{p}$.

Define $w_i^j = \sum_{\alpha_i \in A_i} \overset{j}{w}_{\alpha_i}$. Then $w_i^j \in \Psi_2^p$, and

$$\sum_{j=1}^{N} \sum_{i=1}^{n+1} \overset{j}{w}_i = \sum_{j=1}^{N} \sum_{i=1}^{n+1} \sum_{\alpha_i \in A_i} \overset{j}{w}_{\alpha_i} = \sum_{i=1}^{n+1} \sum_{\alpha_i \in A_i} w_{\alpha_i}$$

$$= \left(\sum_{i=1}^{n+1} \sum_{\alpha_i \in A_i} f_{\alpha_i} \right) w = w. \qquad \text{Q.E.D.}$$

3. LEMMA [3, THEOREM 7.6]. *If* $w \in \Psi_1^p$, $0 \leqq p < n$, *with support contained in the spherical coordinate neighbourhood* U, *then* $w = \sigma - R_g\sigma$, *where* $\sigma \in \Phi_c^p$, supp $\sigma \subset U$, $|g \in G$, $g|_{M-U} = I$.

4. LEMMA. *If* $w \in \Phi^p$ *and* $0 \leqq p < n$, *then* w *is a finite sum of* p*-forms of the form* $\sigma - R_g\sigma$ *with* $\sigma \in \Phi^p$ *and* $g \in G$.

PROOF. By Lemma 2 we may assume $w \in \Psi_2^p$. Then $w = \sum w_i$, $w_i \in \Psi_1^p$, $\{w_i\}$ a disjoint family, supp $w_i \subset U_i$. Then by Lemma 3, $w_i = \sigma_i - R_{g_i}\sigma_i$, where supp $\sigma_i \subset U_i$ and $g_i \in G$, $g_i|_{M-U} = I$.

Let $\sigma = \sum \sigma_i$ and let $g \in G$ be defined by

$$\begin{aligned} g &= g_i \quad \text{on } U_i, \\ &= I \quad \text{on } \bigcap_i (M - U_i). \end{aligned}$$

Then $w = \sigma - R_g\sigma$ and $g \in G$. Q.E.D.

Let Z^p denote the space of closed p-forms, B^p the space of exact p-forms.

5. PROPOSITION. *If* $p < m$, $\mathcal{I}(\Phi^p, Z^0) = 0$.

PROOF. By Lemma 4 above we know that Φ^p is generated by p-forms of the form $\sigma - R_g\sigma$. Let $c \in \mathcal{I}(\Phi^p, Z^0)$. Then

$$\begin{aligned} c(\sigma - R_g\sigma) &= c(\sigma) - cR_\sigma(\sigma) = c(\sigma) - R_gc(\sigma) \\ &= c(\sigma) - c(\sigma) = 0. \qquad \text{Q.E.D.} \end{aligned}$$

6. PROPOSITION. $\mathcal{I}(\Phi^0, \Phi^0) = \boldsymbol{R} \cdot I$.

PROOF. In [3, Theorem 11.2] it is shown that $\mathcal{I}(\Phi^0, \Phi^0) = \boldsymbol{R} \cdot I \oplus \mathcal{I}(\Phi^0, Z^0)$. By Proposition 5 this equals $\boldsymbol{R} \cdot I$. Q.E.D.

7. LEMMA. $\mathcal{I}(Z^1, \Phi^0) = \mathcal{I}(Z^1, Z^0)$.

PROOF. Since $dc \in \mathcal{I}(Z^1, \Phi^1)$ we have $dc = \gamma I$, $\gamma \in \boldsymbol{R}$, by [3, Theorem 12.1]. Now $cd \in \mathcal{I}(\Phi^0, \Phi^0)$. Hence by Proposition 6, $cd = kI$ for some $k \in \boldsymbol{R}$. Let $f \in \Phi^0$. Then $k(df) = d(kf) = dcdf = \gamma(df)$, so that $k = \gamma$, and $cd = \gamma I$. Apply this to $f \equiv 1$ to get $\gamma = 0$. Q.E.D.

8. THEOREM. $\mathcal{I}(\Phi^p, \Phi^0) = 0$ *if* $1 \leqq p < n$.

PROOF. As a result of Proposition 5, it is enough to show that

$$\mathcal{I}(\Phi^p, \Phi^0) = \mathcal{I}(\Phi^p, Z^0).$$

If $p > 1$, this is obvious from [3, Theorem 10.1]. If $p = 1$ and $c \in \mathcal{I}(\Phi^1, \Phi^0)$, then $dc = \gamma I$ by [3, Theorem 10.3]. Suppose $\gamma \neq 0$. Then for $w \in \phi^1$, $w = d(c(1/\gamma)w)$. Thus $\phi^1 = Z^1$. But then Lemma 7 gives the result. Q.E.D.

9. LEMMA. *$\mathcal{J}(Z^p, \Phi^0) \subset (B^p)^0$, where $(B^p)^0 \subset (Z^p)^*$ denotes the annihilator of B^p.*

PROOF. The claim of this lemma is that $c \in \mathcal{J}(Z^p, \Phi^0) \Rightarrow cd = 0$. For $p = 0$ there is nothing to prove. For $p = 1$, the result is contained in the proof of Lemma 7. For $p > 1$, $cd \in \mathcal{J}(\Phi^{p-1}, \Phi^0)$ which is zero by Theorem 8. Q.E.D.

10. THEOREM. *If M is not compact, $\mathcal{J}(\Phi^n, \Phi^0) = 0$.*

PROOF. We have the natural isomorphisms

$$\mathcal{J}(\Phi^n, \Phi^0) = \mathcal{J}(Z^n, \Phi^0) \subset (B^n)^0 \cong (\Phi^n/B^n)^* \cong (H^n)^* = 0$$

where H^n is the nth deRham cohomology group. Q.E.D.

REFERENCES

1. W. Greub and E. Stamm, *On the multiplication of tensor fields*, Proc. Amer. Math. Soc. **17** (1966) 1112–1119. MR **34** #729.

2. L. Jonker, *Natural differential operators*, Ph.D. Thesis, University of Toronto, 1967.

3. R. S. Palais, *Natural operations on differential forms*, Trans. Amer. Math. Soc. **92** (1959), 125–141. MR **22** #7140.

QUEEN'S UNIVERSITY, KINGSTON, ONTARIO, CANADA