SOLVABLE AUTOMORPHISM GROUPS AND
AN UPPER BOUND FOR $|A(G)|$

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Abstract. The objective of this work is to find a subgroup $H$ of a finite group $G$ which will give information about the order of the automorphism group of $G$ and the structure of the automorphism group of $G$. An upper bound is found for the order of the automorphism group and conditions are given which insure that the automorphism group is solvable. Some information is given about a normal subgroup of a particular subgroup of the automorphism group. In this paper all groups are assumed to be finite.

1.0. Notations. $H \triangleleft G$, $H \triangle G$, shall mean respectively, that $H$ is a subgroup, a normal subgroup of a group $G$.

$A(G)$ shall mean the automorphism group of $G$.

$\langle S \rangle =$ subgroup generated by the subset $S$ in $G$.

$N_G(H) =$ the normalizer of the subset $H$ in the group $G$.

$H^x = x^{-1}Hx$ for $x$ in $G$.


$g^\alpha =$ the image of the element $g$ under the mapping $\alpha$.

$H^\alpha =$ the image of the subset under the mapping $\alpha$.

$F(G; H) = \{ \alpha \in A(G) \mid h^\alpha = h$ for each $h$ in $H \}$.

$|G|$ is the number of elements in the group $G$.

$I(G)$ shall mean the inner automorphism group of the group $G$.

1.1. Basic properties of $H$-automorphisms. Hans Liebeck introduced the notation for $H$-automorphism which is adopted.

Definition 1.1.1. Let $H \triangleleft G$. An automorphism $\alpha$ of $G$ is called an $H$-automorphism if $g^{-1}g^\alpha \in H$ for each $g$ in $G$. Denote the set of all $H$-automorphisms by $A(G; H)$.

If $H \triangleleft G$, then $A(G; H) \triangleleft A(G)$. Conversely, any subgroup $T$ of $A(G)$ can be regarded as a group of $H$-automorphisms for various choices of $H$. Since $A(G; G) = A(G)$, one can always choose $H = G$.

Let $T \triangleleft A(G)$ for a group $G$. $K(G; T)$ is used to represent $\langle g^{-1}g^\alpha \mid \alpha \in T, g \in G \rangle$. $K(G; T)$ is called the multiplier subgroup of $T$. Liebeck [3] has shown that $T$ may be a proper subgroup of the group of all $K(G; T)$-automorphisms. One easily shows that $K(G; T) \triangle G$. 

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The set of $\alpha \in A(G)$ such that $H^\alpha = H$ for $H$ a subgroup of $G$ is denoted by $\Gamma(G; H)$. $\Gamma(G; H)$ is clearly a subgroup of $A(G)$ which contains $A(G; H)$. This brings us to

**Proposition 1.1.2.** If $H \leq G$, then $A(G; H) \triangleleft \Gamma(G; H)$.

**Proof.** Let $\alpha \in A(G; H)$ and let $\gamma \in \Gamma(G; H)$. If $g \in G$ then $g^{-1}g\gamma = h\gamma \in H$ for some $h$ in $H$. Therefore $A(G; H) \triangleleft \Gamma(G; H)$.

Since $A(G; H) \triangleleft \Gamma(G; H)$, an obvious question to ask is, "What relationship exists between $\Gamma(G; H)$, $A(G; H)$, and $K(G; A(G; H))$?" The following will give some information in this direction.

**Lemma 1.1.3.** If $H \leq G$, then $K(G; A(G; H))$ is admissible with respect to $\Gamma(G; H)$.

**Proof.** If $\gamma \in \Gamma(G; H)$ and $g^{-1}g^\alpha$ is a generator of $K(G; A(G; H))$, then

$$(g^{-1}g^\alpha)^\gamma = (g^\gamma)^{-1}(g^\gamma)^{-1}g^{-1}g^\alpha \in K(G; A(G; H)).$$

Therefore $K(G; A(G; H))$ is admissible with respect to $\Gamma(G; H)$.

**Theorem 1.1.4.** If $H \leq G$, then there exists a normal subgroup $M$ of $G$ such that $K(G; A(G; H)) \leq M \leq H$, $\Gamma(G; H)/A(G; H)$ is isomorphic to a subgroup of $A(G/M)$, and $M$ is admissible with respect to $\Gamma(G; H)$. $M$ is also maximal with respect to being normal in $G$, being a subgroup of $H$, and being admissible with respect to $\Gamma(G; H)$.

**Proof.** $K(G; A(G; H))$ is normal in $G$, is a subgroup of $H$, and is admissible with respect to $\Gamma(G; H)$. Let $M$ be a subgroup of $G$ maximal with respect to these three properties.

Now define $\phi$ mapping $\Gamma(G; H)$ into $A(G/M)$ by $(g^\sigma)^{(\gamma)^\phi} = (g^\gamma)^\phi$ where $\sigma$ is the natural homomorphism of $G$ onto $G/M$ and $\gamma \in \Gamma(G; H)$. If $g^\sigma$ and $b^\tau$ are two elements of $G/M$, then $(g^\sigma b^\tau)^{\gamma \phi} = (g^\gamma)^{\gamma \phi} (b^\tau)^{\gamma \phi}$.

It then follows that $\gamma \phi$ is an automorphism of $G/M$ for each $\gamma$ in $\Gamma(G; H)$.

For $\gamma$ and $B$ in $\Gamma(G; H)$, $(g^\sigma)^{(\gamma B)^\phi} = (g^\sigma)^{(\gamma)^\phi (B)^\phi}$. It follows from this that $\phi$ is a homomorphism of $\Gamma(G; H)$ into $A(G/M)$.

The kernel of $\phi$ is $A(G; H)$. Therefore $\Gamma(G; H)/A(G; H)$ is isomorphic to a subgroup of $A(G/M)$.

If $H \leq G$, then $\Gamma(G; H)^\alpha = \Gamma(G; H^\alpha)$ and $A(G; H)^\alpha = A(G; H^\alpha)$ for $\alpha$ in $A(G)$.

**1.2 $A$-invariant and $A$-closed subgroups.** Helmut W. Wielandt [4] calls a subgroup $H$ of $G$ intravariant in $G$ if and only if for each automorphism $\alpha$ in $A(G)$ there exists a $g$ in $G$ such that $(H^\alpha)^g = H$. This definition is generalized to give
Definition 1.2.1. A subgroup $H$ of $G$ is $A$-intravariant in $G$ for $A$ a subgroup of $A(G)$ if and only if for each automorphism $\alpha \in A$ there exists a $g$ in $G$ such that $(H^\alpha)^g = H$.

Some of the better known $A(G)$-intravariant subgroups are Sylow subgroups and characteristic subgroups of any group $G$; and Hall subgroups, Carter subgroups, and system normalizers of solvable groups.

Proposition 1.2.2. If $H \leq G$, then $H$ is $A$-intravariant in $G$ for $A$ a subgroup of $A(G)$ if and only if $A \leq \Gamma(G; H)I(G)$.

Proof. Since $I(G) \triangle A(G)$, $\Gamma(G; H)I(G)$ is a well-defined subgroup of $A(G)$. If $H$ is $A$-intravariant then for each $\alpha \in A$ there exist a $g$ in $G$ such that $(H^\alpha)^g = H$. Therefore $\alpha \in \Gamma(G; H)I(G)$ and $A \leq \Gamma(G; H)I(G)$.

If $A \leq \Gamma(G; H)I(G)$ then each element in $A$ may be represented as the product of an element from $\Gamma(G; H)$ and an element from $I(G)$. Therefore $H$ is $A$-intravariant.

Upon considering the subgroup $\Gamma(G; H) \cap I(G)$ for $H \leq G$, it is easily shown that $\Gamma(G; H) \cap I(G) \cong N_\sigma(H)/Z(G)$. This leads to

Proposition 1.2.3. Let $H \leq G$ and assume that $I(G) \leq A \leq A(G)$. Then $H$ is $A$-intravariant if and only if $[G: N_\sigma(H)] = [A: A \cap \Gamma(G; H)]$.

Proof. Since $A \leq \Gamma(G; H)I(G)$,

$$A = A \cap \Gamma(G; H)I(G) = I(G)(A \cap \Gamma(G; H)).$$

Hence

$$|A| = |I(G)| \cdot |A \cap \Gamma(G; H)| / |\Gamma(G; H) \cap I(G)|.$$

Therefore $[A: A \cap \Gamma(G; H)] = [G: N_\sigma(H)]$.

Conversely, suppose that $[G: N_\sigma(H)] = [A: A \cap \Gamma(G; H)]$. Since $I(G) \leq A$, $I(G)(A \cap \Gamma(G; H)) \leq A$. Note that $|A| = |I(G)(A \cap \Gamma(G; H))|$. Hence $A = I(G)(A \cap \Gamma(G; H)) = A \cap \Gamma(G; H)I(G)$. Therefore $A \leq \Gamma(G; H)I(G)$ and $H$ is $A$-intravariant.

It is easy to show that if $H$ is $A$-intravariant in $G$, then $N_\sigma(H)$ is $A$-intravariant in $G$ for $A$ a subgroup of $A(G)$.

In 1937 Philip Hall [2] showed that if $G$ is a solvable group, then the Sylow system $S$ is $A(G)$-intravariant. Then he showed that $A(G) = I(G)\Gamma(G; S)$. With this he was able to arrive at an upper bound for the order of $A(G)$. Later on an upper bound for $|(AG)|$ will be found.

1.3. $A$-closed subgroups. Recall that $H$ is weakly closed in the subgroup $K$ if $H^g \leq K \leq G$ implies that $H^g = H$ for $g$ in $G$.

Definition 1.3.1. A subgroup $H$ of a group $G$ is $A$-closed in $G$ for $A$ a subgroup of $A(G)$ if:
(i) $H$ is $A$-intravariant in $G$, and,
(ii) $H$ is weakly closed in $N_\sigma(H)$.

**Proposition 1.3.2.** If $H \leq G$, then $A(G; H) \leq \bigcap_{\sigma \in \mathcal{G}} \Gamma(G; H^\sigma) \leq A(G; N_\sigma(H))$.

**Proof.** Assume that $\alpha \in A(G; H)$. It then follows that $(h^\sigma)^{-1}(h^\sigma)^\alpha \in K(G; A(G; H)) \triangleq G$. Therefore $K(G; A(G; H)) \leq H^\sigma$ and $(h^\sigma)^\alpha \in H^\sigma$.

Suppose that $\alpha \in \bigcap_{\sigma \in \mathcal{G}} \Gamma(G; H^\sigma)$. If $g \in G$ and $h \in H$, then $h^\sigma = (x^\sigma)^{-1} a$ for some $x$ in $H$. On the other hand, $(x^\sigma)^{-1} a = (x^\sigma) (\sigma)^{-1} = h^{-1}$. Hence $x^\alpha = h^{-1}x^{-1} a$ is an element of $H$ and $g^{-1} g^\alpha \in N_\sigma(H)$. Therefore $\alpha \in A(G; N_\sigma(H))$. It is useful to know when $\bigcap_{\sigma \in \mathcal{G}} \Gamma(G; H^\sigma) = A(G; N_\sigma(H))$. This brings us to

**Proposition 1.3.3.** If $H$ is a $\Gamma(G; N_\sigma(H))$-closed subgroup of $G$, then $A(G; N_\sigma(H)) = \bigcap_{\sigma \in \mathcal{G}} \Gamma(G; H^\sigma)$.

**Proof.** By 1.3.2, $\bigcap_{\sigma \in \mathcal{G}} \Gamma(G; H^\sigma) \leq A(G; N_\sigma(H)) \leq \bigcap_{\sigma \in \mathcal{G}} \Gamma(G; N_\sigma(H)^\sigma)$. It is easily shown that $\Gamma(G; H^\sigma) = \Gamma(G; N_\sigma(H)^\sigma)$ for each $g$ in $G$. Therefore $A(G; N_\sigma(H)) = \bigcap_{\sigma \in \mathcal{G}} \Gamma(G; N_\sigma(H)^\sigma)$.

The converse of 1.3.3. is not true. Let $H$ be the subgroup of the alternating group of degree 5 which is equal to $\langle (1, 2) (3, 4), (1, 3)(2, 4) \rangle$ and it can be shown that $H$ is not weakly closed in $N_\sigma(H)$.

1.4. $A(G)$-closed and $A(G)$-intravariant subgroups. In this section the existence of an $A(G)$-closed subgroup $H$ in the group $G$ will be used to establish the structure of $A(G)$.

**Lemma 1.4.1.** If $H$ is an $A(G)$-intravariant subgroup of $G$, then $A(G)$ is solvable if and only if $\Gamma(G; H)/\Gamma(G; H) \cap I(G)$ and $G$ are solvable.

**Proof.** If $A(G)$ is solvable then it is obvious that $\Gamma(G; H)/\Gamma(G; H) \cap I(G)$ and $G$ are solvable.

Conversely, suppose that $G$ and $\Gamma(G; H)/\Gamma(G; H) \cap I(G)$ are solvable. By 1.2.2 $A(G) = \Gamma(G; H) I(G)$. Hence

$$A(G)/I(G) \cong \Gamma(G; H)/\Gamma(G; H) \cap I(G).$$

Since $G$ is solvable, $I(G)$ is solvable and $A(G)$ is solvable.

**Theorem 1.4.2.** Suppose that $H$ is $A(G)$-closed in the solvable group $G$ with $\Gamma(G; H)$ a maximal subgroup of $A(G)$. If $\Gamma(G; H)/\Gamma(G; H) \cap I(G)$ is solvable with $A(G; N_\sigma(H)) \cong \langle 1 \rangle$, then $A(G)$ is solvable and there exists a minimal normal subgroup $T$ of $A(G)$ and $A(G)$ satisfies the following:

(i) $C_{A(G)}(T) = T$;
(ii) $T$ is the minimal normal subgroup of $A(G)$;
(iii) $A(G) = \Gamma(G; H)T$;
(iv) $T \cap \Gamma(G; H) \cong (1)$;
(v) $|T| = [A(G) : \Gamma(G; H)]$;
(vi) if $G_0/T$ is a minimal normal subgroup of $A(G)/T$ and $A_0 = \Gamma(G; H) \cap G_0$, then $\Gamma(G; H) = N_{A(G)}(A_0)$, $|G_0| = |T||A_0|$, $|T| = p^a$, $|A_0| = q^b$ and $p \neq q$ for primes $p$ and $q$ and nonnegative integers $a$ and $b$.

**Proof.** By 1.4.1, $A(G)$ is a solvable group. Hence $A(G)$ has a minimal normal subgroup $T$.

Note that $\Gamma(G; H) = \bigcap_{\alpha \in A(G)} \Gamma(G; H)$. It follows from [4] that $A(G) = \Gamma(G; H)T$ where $T$ is the minimal normal subgroup of $G$. Then $C_{A(G)}(T) = T$, $\Gamma(G; H) \cap T \cong (1)$, $[A(G) : \Gamma(G; H)] = T$, and condition (vi) is satisfied.

The condition that $\Gamma(G; H)$ be a maximal subgroup of $A(G)$ could be replaced by $[G: N_0(H)]$ is a prime number. Then $|T| = [G: N_0(H)]$.

Continuing in the search for groups whose automorphism groups are solvable we arrive at

**Proposition 1.4.3.** Let each subgroup in a composition series of the subgroup $H$ of the solvable group $G$ be $T(G; H)$-closed for $H$ an $A(G)$-intravariant subgroup of $G$. If $F(G; H)$ is solvable then $A(G)$ is solvable.

**Proof.** By 1.4.1, $A(G)$ is solvable if $\Gamma(G; H)/T(G; H) \triangleleft A(G)$ is solvable. Note that $F(G; H) \triangleleft A(G)$.

Let $1 = N_r \leq N_{r-1} \leq \cdots \leq N_0 = H$ be the composition series of $H$ mentioned in the hypothesis. Using induction it is easy to show that each $N_i \triangle H$ for $i = 0, 1, \cdots, r$. If $\alpha \in \Gamma(G; H)$, then $N_i^\alpha = N_i \leq H$ for some $g$ in $G$. Since $N_i^\alpha \leq H \leq N_0(H)$ we have that $N_i^\alpha = N_i$. Therefore $\Gamma(G; H)$ leaves each subgroup in the series fixed. Note that $\Gamma(G; H)/F(G; H)$ is isomorphic to a subgroup of $A(H)$. By Gaschütz [1], $\Gamma(G; H)/F(G; H)$ is solvable, $\Gamma(G; H)$ is solvable and the proof is complete.

1.5. **An upper bound for the order of $A(G)$**. An upper bound for $|A(G)|$ will be derived by using the concept of $H$ being $A(G)$-intravariant in $G$.

**Theorem 1.5.1.** Let $H$ be an $A(G)$-intravariant subgroup of $G$ with $F(G; K_1) \leq A(G; H)$ where $K_1 = K(G; A(G; H))$. Then $|A(G)|$ divides $[G: N_0(H)]|A(K_1)| |F(G; K_1)|$. The prime divisors of $|F(G; K_1)|$ are prime divisors of $|K_1|$.

**Proof.** If $p$ is a prime dividing $|F(G; K_1)|$ then there exists an $\alpha$ in $F(G; K_1)$ such that $|\alpha| = p$. Since $\alpha$ is not the identity, there exists a $g$...
in $G$ such that $g^{-1}g^a = k \in K_1$, $k \neq 1$. Hence $g^{a^p} = gk^p = g$. Therefore $k^p = 1$ and $p$ divides $|K_1|$.

By 1.1.3, $K_1$ is admissible with respect to $\Gamma(G; H)$. Define $\phi$ mapping $\Gamma(G; H)$ into $A(K_1)$ by $(k^{\gamma})^{\phi} = k^\gamma$ for each $k$ in $K_1$. $\phi$ is a homomorphism of $\Gamma(G; H)$ into $A(K_1)$ and the kernel of $\phi$ is $F(G; K_1)$. Therefore $\Gamma(G; H)/F(G; K_1)$ is isomorphic to a subgroup of $A(K_1)$.

It follows from 1.2.3 that $[A(G):\Gamma(G; H)] = [G:N_G(H)]$. Since $|\Gamma(G; H)/F(G; K_1)|$ divides $|A(K_1)|$, $|A(G)|$ divides $[G:N_G(H)]$ $|A(K_1)||F(G; K_1)|$.

We are now in a position to prove

**Corollary 1.5.2.** Let $H$ be a characteristic subgroup of $G$ such that $F(G; K_1) \leq A(G; H)$ where $K_1 = K(G; A(G; H))$ is a cyclic $p$-group of $G$, then $|A(G)|$ divides $p^m(p - 1)$ for some nonnegative integer $m$.

**Proof.** If $|K_1| = 2^t$ then $|A(G)|$ divides some power of 2.

Assume that the order of $K_1$ is equal to $p^t$ for an odd prime $p$. Then $|A(K_1)| = p^{t-1}(p - 1)$ and $|A(G)|$ divides $p^m(p - 1)$ for some nonnegative integer $m$.

**Corollary 1.5.3.** Let $H$ be a characteristic subgroup of $G$ such that $K_1 = K(G; A(G; H))$ is a cyclic $p$-group with $p$ a prime of the form $2^m + 1$ for some integer $m$. If $F(G; K_1) \leq A(G; H)$ then $|A(G)|$ divides $p^{n2^m}$, $A(G)$ is solvable, and $G$ is solvable.

**Proof.** By 1.5.2, $|A(G)|$ divides $p^{n2^m}$ for some integer $n$. Hence $A(G)$ and $G$ are solvable.

**Corollary 1.5.4.** If $H$ is a normal Sylow 2-subgroup of $G$ such that $F(G; K_1) \leq A(G; H)$ for $K_1 = K(G; A(G; H))$ cyclic, then $G = H$.

**Proof.** By 1.5.2, $|A(G)| = 2^n$ for some integer $n$. Hence $A(G)$ is a nilpotent group and $G$ is nilpotent.

Assume that there exists a prime $p \neq 2$ which divides $|G|$. Since $G$ is a nilpotent group, $A(G)$, is a direct product of the automorphism groups of the Sylow subgroups of $G$.

Let $\alpha \in A(H_p)$ where $H_p$ is the Sylow $p$-subgroup of $G$. Note that $\alpha$ can be thought of as an element in $A(G)$ that acts as the identity on $H$ and is an element of $A(G; H)$. If $g \in H_p$ then $g^{-1}g^a \in H \cap H_p = \langle 1 \rangle$. Therefore $\alpha$ is the identity on $H_p$ and $A(H_p) \cong \langle 1 \rangle$. Since $2 < p$, we have a contradiction. Therefore $H = G$.

The upper bound for $|A(G)|$ found in 1.5.1 is actually attained if $G$ is the symmetric group of degree 3 and $H$ is the alternating group of degree 3.
References


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