A NOTE ON THE LOCATION OF CRITICAL POINTS OF POLYNOMIALS

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Abstract. Let \( \mathcal{P}(a, 3) \) denote the set of cubic polynomials which have all of their zeros in \( |z| \leq 1 \) and at least one zero at \( z = a \) \((|a| \leq 1)\). In this paper we describe a minimal region \( \mathcal{D}(a, 3) \) with the property that every polynomial in \( \mathcal{P}(a, 3) \) has at least one critical point in \( \mathcal{D}(a, 3) \). The location of the zeros of the logarithmic derivative of the function \((z - a)^m(z - z_1)^m_1(z - z_2)^m_2\) is also discussed.

1. Introduction. Let \( p(z) \) be a polynomial of degree \( n (\geq 2) \) having all its zeros in the closed disk \( \gamma: |z| \leq 1 \). Ilieff has conjectured \([1, p. 25]\) that if \( a \) is a zero of \( p(z) \), then at least one critical point of \( p(z) \) (i.e., zero of \( p'(z) \)) lies in the disk \( |z - a| \leq 1 \). A more difficult problem related to this conjecture is stated in \([2]\) as follows:

Let \( \mathcal{P}(a, n) \) be the set of all \( n \)th degree polynomials which have all of their zeros in \( \gamma \) and at least one zero at the point \( z = a \). Describe a region \( \mathcal{D}(a, n) \) such that (i) \( \mathcal{D}(a, n) \) contains at least one critical point of every \( p(z) \in \mathcal{P}(a, n) \) and such that (ii) no proper subset of \( \mathcal{D}(a, n) \) has property (i).

It is the aim of the present note to describe sets \( \mathcal{D}(a, 3) \), \(|a| \leq 1\), and thereby improve the results of Schmeisser \([3]\) and others \([4], [5]\) concerning the location of critical points of cubic polynomials. We shall define only those sets \( \mathcal{D}(a, 3) \) for which \( 0 \leq a \leq 1 \), since the remaining sets can then be obtained by rotation. Our main result is

**Theorem 1.** Let \( p(z) = (z - a)(z - z_1)(z - z_2) \), where \( 0 \leq a \leq 1 \), \(|z_1| \leq 1\), and \(|z_2| \leq 1\). Let \( \Delta(a) \) denote the closed disk

\[
\Delta(a): |z - a/2| \leq ((4 - a^2)/12)^{1/2},
\]

and \( C(a) \) denote its circumference. Then \( p'(z) \) has at least one zero in the set \( \mathcal{D}(a, 3) \) defined by

\[
\mathcal{D}(a, 3) = \Delta(a) \setminus [C(a) \cap \{z: \text{Im} z > 0\}], \quad a > 0,
\]

\[
\mathcal{D}(0, 3) = \Delta(0).
\]
Furthermore if $D(a, 3)$ is replaced by any proper subset of $D(a, 3)$, then the assertion is false.

The proof of Theorem 1 is given in §2. In §3 we consider the location of the zeros of the logarithmic derivative of the function $(z-a)^m(z-z_1)^m(z-z_2)^m$, and in §4 we mention an open problem that is suggested by our results.

2. **Proof of Theorem 1.** The proof is based upon the following simple lemma:

**Lemma 1.** If $f(z) = b_1 z^2 + b_1 z + b_0$ is nonzero in $|z| < 1$, then

\[ |b_0|^2 - |b_2|^2 \geq |b_1 b_2 - b_0 b_1|. \]

**Proof.** The case $b_2 = 0$ is trivial. The case $b_2 \neq 0$ is at once reduced to $b_2 = 1$.

Let $\alpha, \beta$ be the roots of $\xi^2 + b_1 \xi + b_0 = 0$, $|\alpha| \leq 1$, $|\beta| \leq 1$. Then $(|\alpha| - 1)(|\beta| - 1) \geq 0$, and so $|\alpha \beta| + 1 \geq |\alpha| + |\beta|$. Multiplying both sides of the last inequality by $|\alpha \beta| - 1 = |b_0| - 1 (\geq 0)$ there follows

\[ |b_0|^2 - 1 \geq |\alpha|^2 |\beta| + |\alpha| |\beta|^2 - |\alpha| - |\beta| \]

\[ = (|\alpha|^2 - 1) |\beta| + (|\beta|^2 - 1) |\alpha| \]

\[ \geq (|\alpha|^2 - 1) |\beta| + (|\beta|^2 - 1) |\alpha| = |b_1 - b_0 b_1|, \]

which proves Lemma 1.

We can now prove

**Lemma 2.** Let $p(z) = (z-a)(z-z_1)(z-z_2)$, where $0 < a \leq 1$, $|z_1| = 1$, and $|z_2| \leq 1$. If $p'(z)$ has no zero inside $C(a)$, then $z_2 = z_1$ or $a = z_1 = 1$ or $a = z_2 = 1$.

Furthermore, if $p_a(z) = (z-a)(z-e^{ia})(z-e^{-ia})$ and

\[ 0 \leq \alpha_1 \equiv \cos^{-1} \left( \frac{a + 6A}{4} \right) \leq \alpha \leq \cos^{-1} \left( \frac{a - 6A}{4} \right) \equiv \alpha_2 \leq \pi, \]

where $A \equiv ((4-a^2)/12)^{1/2}$, then $p_a'(z)$ has a pair of conjugate zeros and as $\alpha$ varies from $\alpha_1$ to $\alpha_2$ these zeros describe $C(a)$. If, however, $0 \leq \alpha < \alpha_1$ or $\alpha_2 < \alpha \leq \pi$, then $p_a'(z)$ has a zero inside $C(a)$.

**Proof.** Let $z_1 = \exp[\theta_1]$ and $z_2 = r \exp[\theta_2]$ ($0 \leq r \leq 1$). To prove the first part of the lemma we note initially that since $p'(z)$ is nonzero inside $C(a)$, the polynomial

\[ P(\xi) = p'(A \xi + a/2) = 3A^2 \xi^2 + A(a - 2z_1 - 2z_2) \xi + z_1 z_2 - a^2/4 \]

is nonzero in $|\xi| < 1$. Hence as a consequence of (1) we have

\[ |b_0|^2 - |b_2|^2 \geq \Re(b_1 b_2 - b_0 b_1), \]

where $a \equiv ((4-a^2)/12)^{1/2}$, then $p_a'(z)$ has a pair of conjugate zeros and as $\alpha$ varies from $\alpha_1$ to $\alpha_2$ these zeros describe $C(a)$. If, however, $0 \leq \alpha < \alpha_1$ or $\alpha_2 < \alpha \leq \pi$, then $p_a'(z)$ has a zero inside $C(a)$.
where \( b_k \) denotes the coefficient of \( \xi^k \) in the expansion of \( P(\xi) \). Since
\[
\text{Re}(b_1b_2 - b_0b_1) = A\left[a(1 - r \cos(\theta_1 + \theta_2)) - 2(1 - r^2) \cos \theta_1\right]
\]
and \( A \geq a/2 \), we deduce from (2) that
\[
a^2(1 - r \cos(\theta_1 + \theta_2))/2 - (1 - r^2)
\]
\[
\geq a^2(1 - r \cos(\theta_1 + \theta_2))/2 - a \cos \theta_1(1 - r^2),
\]
that is \((1-r^2)(a \cos \theta_1 - 1) \geq 0\). Hence either \( r = 1 \) or \( 1 = a = \cos \theta_1 = z_1 \).
If \( r = 1 \), then (2) implies that
\[
(a/2 - A)(1 - \cos(\theta_1 + \theta_2)) \
\geq 0,
\]
and so either \( a/2 = A \), i.e., \( a = 1 \), or \( \cos(\theta_1 + \theta_2) = 1 \), i.e., \( z_2 = z_1 \). Finally, if \( r = 1 \) and \( a = 1 \), we have equality in (2) and it follows from (1) that
\[
\text{Im}(b_1b_2 - b_0b_1) = \sin(\theta_1 + \theta_2) - \sin \theta_1 - \sin \theta_2 = 0.
\]
This last equation implies that \( \theta_1 = 0 \) or \( \theta_2 = 0 \) or \( \theta_1 + \theta_2 \equiv 0 \) (mod 2\( \pi \)), i.e., \( z_1 = 1 \) or \( z_2 = 1 \) or \( z_2 = z_1 \). The proof of the first part of Lemma 2 is now complete.

To prove the second part we note that \( p'(z) = (A \xi + a/2) \) has the zeros
\[
\xi_1, \xi_2 = -\left(\frac{a - 4 \cos \alpha}{2} \pm \sqrt{\left(\frac{a - 4 \cos \alpha}{2}\right)^2 - 36A^2}\right)
\]
and that the quantity under the radical is nonpositive if \( \alpha_1 \leq \alpha \leq \alpha_2 \). Furthermore, it is easily verified that as \( \alpha \) varies from \( \alpha_1 \) to \( \alpha_2 \) the points \( \xi_1, \xi_2 \) describe the boundary of the unit disk. Finally, if \( 0 \leq \alpha < \alpha_1 \) or \( \alpha < \alpha \leq \pi \), then \( \xi_1 \) and \( \xi_2 \) are real and unequal and, since their product is unity, it follows that one of them lies in \( |\xi| < 1 \). This implies the second part of Lemma 2 and concludes the proof.

We next establish

**Lemma 3.** Let \( p(z) = (z-a)(z-z_1)(z-z_2) \), where \( 0 \leq a \leq 1 \), \( |z_1| < 1 \), and \( |z_2| < 1 \). Then \( p'(z) \) has at least one zero inside \( C(a) \).

**Proof.** Since \( |z_1| < 1 \) and \( |z_2| < 1 \), there exists a number \( \rho \) (0 < \( \rho < 1 \)) such that
\[
q(w) = p(\rho w + (1 - \rho)a) = \rho^3(w - a)(w - w_1)(w - w_2),
\]
where \( |w_1| = 1 \) and \( |w_2| \leq 1 \). By Lemma 2, \( q'(w) \) has a zero in \( |w - a/2| \leq A \) and hence \( p'(z) \) has a zero (\( \neq 1 \)) in the disk
\[
\gamma(\rho, a) : |z - a/2 - (1 - \rho)a/2| \leq \rho A.
\]
Since \( A \geq a/2 \) with equality only for \( a = 1 \), it is readily verified that
every point of $\gamma(\rho, a)$, except $z = 1$ in the case $a = 1$, lies inside $C(a)$. This proves the lemma.

Combining Lemmas 2 and 3 we clearly obtain the first part of Theorem 1 for $a > 0$. For $a = 0$ the required result is an immediate consequence of the fact that the modulus of the product of the zeros of $p'(z)$ is less than or equal to $1/3$.

To prove the second part of Theorem 1 for the case $a > 0$ it suffices to show that for any point $z_0$ inside $C(a)$ there exists $p(z) \in \Phi(P, 3)$ such that $p'(z_0) = 0$ and $p'(z) \neq 0$ for all $z \neq z_0$ in $D(a, 3)$. Consider first the polynomial

$$p(z, t) = (z - a)(z - e^{it})(z - \beta(t)),$$

where $\beta(t) = (1-t)a + te^{it}$ ($0 \leq t \leq 1$). For $t = 0$ we have $p(z, 0) = (z - a)^2(z - e^{it})$ so that $p'(z, 0)$ has a zero at $a$ and at $(a + 2e^{it})/3$. It is easy to see that the second zero lies outside $C(a)$ except for the trivial case $a = e^{it} = 1$. As $t$ varies continuously from 0 to 1, each root of $p'(z, t)$ varies continuously on the line segment $[a, e^{it}]$, one from $a$ to $(2a + e^{it})/3$, the other from $(a + 2e^{it})/3$ to $e^{it}$ (compare [6, p. 24]). It follows, therefore, that no point $z_0$ of the closed disk $|z_0 - 2a/3| \leq 1/3$ can be omitted from $D(a, 3)$.

Suppose next that $z_0$ is a point inside $C(a)$ for which $|z_0 - 2a/3| > 1/3$, and choose $z_0$ ($|z_0| > 1$) such that the derivative of $P_0(z) = (z - a)(z - z_0^2)$ has a zero at $z_0$. We note that

$$\frac{P_0'(z_0)}{P_0(z_0)} = \frac{1}{z_0 - a} + \frac{2}{z_0 - z_0} = 0.$$  

Now consider the mapping $l(z) = 1/(z_0 - z)$. Under $w = l(z)$ the unit circumference is mapped onto a circle $\Gamma$ and $z_0$ is mapped to a point $w_0$ inside $\Gamma$. Choose $w_1$, $w_2$ on $\Gamma$ such that $w_0 = (w_1 + w_2)/2$, and let $z_1$, $z_2$ be the points on $|z| = 1$ that satisfy $w_1 = l(z_1)$, $w_2 = l(z_2)$. It follows that

$$\frac{2}{z_0 - z_0'} = \frac{1}{z_0 - z_1'} + \frac{1}{z_0 - z_2'},$$

and, combining (3) and (4), we deduce that the derivative of $p_0(z) \equiv (z - a)(z - z_1')(z - z_2')$ has a zero at $z = z_0$. Since $|z_1'| = |z_2'| = 1$ implies that $|z_1'z_2' - a^2/4| \geq 3A^2$, it is easily seen by considering the polynomial $p_0'(A\xi + a/2)$ that the second zero of $p_0'(z)$ lies outside $C(a)$. This proves Theorem 1 for $a > 0$.

For the case $a = 0$ we observe that the derivative of

$$z(z - re^{i\alpha})(z - re^{(\alpha + r)/3}), \quad 0 \leq \alpha \leq 2\pi, \quad 0 \leq r \leq 1,$$
has a double zero at $z = re^{i(a + \pi/6)}/\sqrt{3}$, and hence no point may be omitted from $\mathcal{D}(0, 3)$. This completes the proof of Theorem 1.

3. The logarithmic derivative. By applying the methods of §2 it is possible to prove

**Theorem 2.** Let $m, m_1, \text{ and } m_2$ be nonnegative real numbers with $n = m + m_1 + m_2 > 0$, and suppose

$$F(z) = \frac{m}{z - a} + \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2},$$

where $0 \leq a \leq 1$, $|z_1| \leq 1$, and $|z_2| \leq 1$. Set

$$\alpha(m, n) = \frac{(n - m)a/(n + m)}{A(m, n) = (m(1 - \alpha(m, n))^2)/n}^{1/2}.$$

Then $F(z)$ has at least one zero in the disk

$$(5) |z - \alpha(m, n)| \leq A(m, n).$$

We note that if $p(z)$ is a polynomial of the form

$$p(z) = (z - a)^m(z - z_1)^{m_1}(z - z_2)^{m_2},$$

where $0 \leq a \leq 1$, $|z_1| \leq 1$, and $|z_2| \leq 1$, then Theorem 2 implies that $p(z)$ has at least one critical point distinct from $a$ in the disk (5). It follows easily from this that Ilieff's conjecture is true for such $p(z)$.

4. Conjecture. The equation

$$F(z) = \frac{m}{z - a} + \sum_{k=1}^{s} \frac{m_k}{z - z_k} = 0$$

$(m \geq 0, m_k \geq 0, n = m + m_1 + \cdots + m_s > 0, 0 \leq a \leq 1, |z_k| \leq 1 (1 \leq k \leq s))$ has a root $\xi$ satisfying

$$|\xi - \alpha| \leq \left\{(1 - \alpha)^{s-1} \left(\frac{m}{n} + \frac{\alpha(s - 1)m}{n}\right)\right\}^{1/s},$$

where $\alpha = (n - m)a/(n + (s - 1)m)$.

Support for the conjecture comes from Theorem 2 (the case $s = 2$). It is also easy to verify that the conjecture is true for $a = 0$. Its truth for $a = 1$ follows from a modification of the proof in [2].

We remark that for the case where $m = 1, m_k = 1 (1 \leq k \leq s)$, the conjecture is due to J. S. Ratti and is sharper than Ilieff's.

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References


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