ON THE IDEAL OF UNCONDITIONALLY CONVERGENT FOURIER SERIES IN $L_p(G)$

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Abstract. Let $G$ be a compact abelian group. We consider the ideal of functions in $L_p(G)$ with unconditionally convergent Fourier series in the $L_p$ norm. This ideal is shown to coincide with the "Derived algebra" of Helgason. A characterization of this ideal is given when $p$ is an even integer.

Let $G$ be a compact abelian group with dual group $\Gamma$ and with normalized Haar measure (so that $f \mapsto \hat{f}$ is an isometry of $L_1(G)$ onto $l_1(\Gamma)$). For $1 \leq p \leq \infty$, $L_p(G)$ is a Banach algebra with convolution for multiplication. Let $S_p$ denote the ideal of functions in $L_p$ having unconditionally convergent Fourier series in the $L_p$ norm.

In §2 we show that, for the Banach algebra $L_p$, $S_p$ coincides with the "Derived algebra" of Helgason [7]; from this and [7] one immediately obtains that $S_p = L_2$ for $1 \leq p < 2$. This is a special case of a theorem of Grothendieck [4].

In §3 we show that, for $p$ an even integer, if $f \in L_p$ and $\hat{f} \geq 0$ then then $f \in S_p$. From this it follows that (for $p$ an even integer) $S_p$ coincides with the set of $f \in L_p$ such that $|\hat{f}|$ is the transform of an $L_p$ function. Thus for such $p$ one can characterize the Rudin $\Lambda(p)$ sets as those subsets $E \subseteq \Gamma$ for which $|\hat{f}|$ is the transform of an $L_p$ function whenever $f \in L_p$ and $\hat{f} = 0$ outside of $E$.

1. Preliminaries. For facts we use about harmonic analysis the reader is referred to [3] and [9], for unconditional convergence and related notions, to [2], and for $L_p(G)$ and dual algebras, to [8].

Let $\mathcal{S}$ denote the set of finite subsets of $\Gamma$.

Theorem 1. Let $1 \leq p \leq \infty$. Then $S_p$ is a semisimple dual Banach algebra with the norm given by

$$\|f\|_{S_p} = \sup_{J \in \mathcal{S}} \left\| \sum_{\gamma \in J} \hat{f}(\gamma) \gamma \right\|_p.$$ 

If $p = \infty$, then $S_p$ is isomorphic and homeomorphic to $l_1(\Gamma)$, where the latter has pointwise multiplication.
Proof. (See [1] for details.) One verifies that $S_p$ is a Banach algebra and that each $f \in S_p$ has an unconditionally convergent Fourier series in the $S_p$ norm. Thus each closed ideal in $S_p$ is the closed linear span of the characters it contains. From this it follows that $S_p$ is a semisimple dual algebra.

The second assertion is straightforward to verify.

2. $S_p$ and the Derived algebra.

Definition (Helgason). For $1 \leq p < \infty$, define the Derived algebra, $D_p$, to be the set of $f \in L_p$ such that

$$\sup_{g \in L_p} \frac{||f \ast g||_p}{||g||_\infty} \leq ||f||_{D_p} < \infty.$$ 

After the following lemma, we will show that $D_p$ and $S_p$ are the same Banach algebra.

Lemma 1. Every element of $l_\infty(\Gamma)$ is a multiplier for $S_p$.

Proof. We want to show that, given $f \in S_p$, $a \in l_\infty(\Gamma)$, there exists $g \in S_p$ such that $g = a f$. This follows from the fact that unconditional convergence implies bounded-multiplier convergence in a Banach space [2, p. 59], [5, p. 92].

Theorem 2. The Banach algebras $S_p$ and $D_p$ coincide, and $|| \cdot ||_{S_p} \leq || \cdot ||_{D_p}$.

Proof. By Theorem 2 of [7], $D_p$ is the set of functions in $L_p$ for which every element of $c_0(\Gamma)$ is a multiplier. Thus by the lemma, $S_p \subseteq D_p$.

If $f \in L_p$, $\gamma \in \mathcal{F}$, and $g = \sum_j f(\gamma_j) \gamma$, then $||g||_\infty = 1$ and $f \ast g = \sum_j f(\gamma_j) \gamma_j$. Thus $D_p \subseteq S_p$ and $|| \cdot ||_{S_p} \leq || \cdot ||_{D_p}$. By the Closed Graph Theorem, the two norms are equivalent.

Corollary (Grothendieck). Let $1 \leq p < 2$. If $f \in L_p$ and $f$ has an unconditionally convergent Fourier series in the $L_p$ norm, then $f \in L_2$.

Proof. This follows directly from the above theorem and [7, Theorems 5 and 7].

3. A characterization of $S_p$ and the Rudin $\Lambda(p)$ sets when $p$ is an even integer. We first prove the following about functions with non-negative transform:

Theorem 3. Let $p = 2k$, $k$ an integer, and let $f \in L_p$ such that $\hat{f} \geq 0$. 

1This follows from the Orlicz-Pettis Theorem [2, p. 60].
Then

1. \( f \) has an unconditionally convergent Fourier series in the \( L_p \) norm;
2. \( \|f\|_p = \|f\|_{D_p} = \|f\|_{S_p} \).

**Proof.** We claim that it is sufficient to show that

\[
\sup \{ \|f \ast g\|_p \mid g \text{ a trigonometric polynomial, } \|g\|_\infty \leq 1 \} \leq \|f\|_p.
\]

Since the trigonometric polynomials are dense in \( L_p \), we then deduce

\[
\|f \ast h\|_p \leq \|f\|_p \|h\|_\infty, \quad h \in L_p.
\]

Thus \( f \in D_p = S_p \), so (1) follows. Now the above inequality gives that

\[
\|f\|_{D_p} \leq \|f\|_p. \quad \text{Since } \|f\|_p \leq \|f\|_S \leq \|f\|_{D_p}, \quad (2) \text{ follows as well.}
\]

So, let \( g = \sum_{i=1}^n a_i \gamma_i \), where \( |a_i| \leq 1 \), and let

\[
h = f \ast g = \sum_{i=1}^n \hat{f}(\gamma_i) a_i \gamma_i.
\]

Letting \("[k]\) denote \( k \)th convolution power, we have that

\[
\|h\|_p^2 = \int |h|^2 = \|h^{[k]}\|_2^2 = \|f^{[k]}\|_2^2 \leq \|f\|_2^2 = \|f\|_p^2.
\]

Since \( |h| \leq f \), we have that

\[
\|h^{[k]}\|_2^2 \leq \|f^{[k]}\|_2^2 = \|f^{[k]}\|_2^2 = \int |f|^{2k} = \|f\|_p^2.
\]

Therefore \( \|h\|_p \leq \|f\|_p \), and the theorem is established.

As a corollary, we have the following characterization of \( S_p \):

**Theorem 4.** Let \( p \) be an even integer, and let \( f \in L_p \). Then the following statements are equivalent:

1. \( f \) has an unconditionally convergent Fourier series in the \( L_p \) norm.
2. \( |f| \) is the Fourier transform of an \( L_p \) function.

**Proof.** Lemma 1 gives that (1) implies (2). If (2) holds and \( g = |f| \), then \( g \in S_p \) by the above theorem; so \( f \in S_p \) by Lemma 1.

**Remarks.** (1) For \( G \) a compact abelian group, Bochner's theorem [9, p. 19] may be stated as follows:

If \( \phi \) is continuous on \( G \) and \( \phi \geq 0 \), then \( \phi \in S_\infty \). Thus Theorem 3 is an \( L_p \) analog of Bochner's theorem, for \( p \) an even integer.

(2) The essential ingredient in the proof of Theorem 3 is the fact that, for \( p \) an even integer, if one multiplies the coefficients of a trigonometric polynomial by complex numbers of absolute value one, the largest \( L_p \) norm is obtained when the coefficients become non-negative. This was first observed by Hardy and Littlewood [6, p.
It is an open question as to what the maximum is for other values of $p$.

For $E \subseteq \Gamma$, $1 < p < \infty$, let $L_p^E$ be the set of functions in $L_p(G)$ whose transforms vanish outside of $E$. The set $E$ is said to be of type $\Lambda(p)$ if there exists $q < p$ such that $L_q^E = L_p^E$. For a discussion of equivalent notions, the reader is referred to [3].

The following lemma is known.

**Lemma 2.** Let $2 < p < \infty$. Then $E$ is of type $\Lambda(p)$ if and only if $L_p^E \subseteq S_p$.

**Proof.** If $E$ is of type $\Lambda(p)$, then $L_p^E = L_2^E$, so $L_p^E \subseteq S_p$.

Suppose conversely that $L_p^E \subseteq S_p$. Let $f \in L_2^E$. Then (cf. [3, 14.3.2]) for some $a : \Gamma \to \{-1, 1\}$, the function $af$ is the transform of a function in $L_2^E$. By Lemma 1, $a(af) = f$ is also the transform of an $L_p^E$ function. Therefore $L_p^E = L_2^E$, so $E$ is of type $\Lambda(p)$.

As a direct consequence of the above lemma and Theorem 4, we have the following characterization of sets of type $\Lambda(p)$:

**Theorem 5.** Let $p > 2$ be an even integer and let $E \subseteq \Gamma$. Then $E$ is of type $\Lambda(p)$ if and only if $|f|$ is the transform of an $L_p^E$ function whenever $f \in L_p^E$.

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**References**


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