ON THE IDEAL OF UNCONDITIONALLY CONVERGENT
FOURIER SERIES IN $L_p(G)$

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Abstract. Let $G$ be a compact abelian group. We consider the
ideal of functions in $L_p(G)$ with unconditionally convergent Fourier
series in the $L_p$ norm. This ideal is shown to coincide with the "De-
rived algebra" of Helgason. A characterization of this ideal is given
when $p$ is an even integer.

Let $G$ be a compact abelian group with dual group $\Gamma$ and with
normalized Haar measure (so that $f \rightarrow \hat{f}$ is an isometry of $L_2(G)$ onto
$L_2(\Gamma)$). For $1 \leq p \leq \infty$, $L_p(G)$ is a Banach algebra with convolution for
multiplication. Let $S_p$ denote the ideal of functions in $L_p$ having
unconditionally convergent Fourier series in the $L_p$ norm.

In §2 we show that, for the Banach algebra $L_p$, $S_p$ coincides with
the "Derived algebra" of Helgason [7]; from this and [7] one imme-
diately obtains that $S_p=L_2$ for $1 \leq p < 2$. This is a special case of a
theorem of Grothendieck [4].

In §3 we show that, for $p$ an even integer, if $f \in L_p$ and $\hat{f} \equiv 0$ then
then $f \in S_p$. From this it follows that (for $p$ an even integer) $S_p$
coincides with the set of $f \in L_p$ such that $|\hat{f}|$ is the transform of an
$L_p$ function. Thus for such $p$ one can characterize the Rudin $\Lambda(p)$
sets as those subsets $E \subset \Gamma$ for which $|\hat{f}|$ is the transform of an $L_p$
function whenever $f \in L_p$ and $\hat{f} = 0$ outside of $E$.

1. Preliminaries. For facts we use about harmonic analysis the
reader is referred to [3] and [9], for unconditional convergence and
related notions, to [2], and for $L_p(G)$ and dual algebras, to [8].

Let $\mathcal{S}$ denote the set of finite subsets of $\Gamma$.

Theorem 1. Let $1 \leq p \leq \infty$. Then $S_p$ is a semisimple dual Banach
algebra with the norm given by

$$||f||_{S_p} = \sup_{\mathcal{S}} \left\| \sum_j \hat{f}(\gamma) \gamma \right\|_p.$$

If $p = \infty$, then $S_p$ is isomorphic and homeomorphic to $l_1(\Gamma)$, where the
latter has pointwise multiplication.
PROOF. (See [1] for details.) One verifies that $S_p$ is a Banach algebra and that each $f \in S_p$ has an unconditionally convergent Fourier series in the $S_p$ norm. Thus each closed ideal in $S_p$ is the closed linear span of the characters it contains. From this it follows that $S_p$ is a semisimple dual algebra.

The second assertion is straightforward to verify.

2. $S_p$ and the Derived algebra.

Definition (Helgason). For $1 \leq p < \infty$, define the Derived algebra, $D_p$, to be the set of $f \in L_p$ such that

$$\sup_{g \in L_p} \frac{||f \ast g||_p}{||g||_\infty} = ||f||_{D_p} < \infty.$$ 

After the following lemma, we will show that $D_p$ and $S_p$ are the same Banach algebra.

Lemma 1. Every element of $l_\infty(\Gamma)$ is a multiplier for $S_p$.

Proof. We want to show that, given $f \in S_p$, $a \in l_\infty(\Gamma)$, there exists $g \in S_p$ such that $g = af$. This follows from the fact that unconditional convergence implies bounded-multiplier convergence in a Banach space [2, p. 59], [5, p. 92].

Theorem 2. The Banach algebras $S_p$ and $D_p$ coincide, and $|| \cdot ||_{S_p} \leq || \cdot ||_{D_p}$.

Proof. By Theorem 2 of [7], $D_p$ is the set of functions in $L_p$ for which every element of $c_0(\Gamma)$ is a multiplier. Thus by the lemma, $S_p \subseteq D_p$.

If $f \in L_p$, $J \in \mathcal{F}$, and $g = \sum_j \gamma_j$, then $||g||_\infty = 1$ and $f \ast g = \sum_j f(\gamma_j)\gamma_j$. Thus $D_p \subseteq S_p$ and $|| \cdot ||_{S_p} \leq || \cdot ||_{D_p}$. By the Closed Graph Theorem, the two norms are equivalent.

Corollary (Grothendieck). Let $1 \leq p < 2$. If $f \in L_p$ and $f$ has an unconditionally convergent Fourier series in the $L_p$ norm, then $f \in L_2$.

Proof. This follows directly from the above theorem and [7, Theorems 5 and 7].

3. A characterization of $S_p$ and the Rudin $\Lambda(p)$ sets when $p$ is an even integer. We first prove the following about functions with non-negative transform:

Theorem 3. Let $p = 2k$, $k$ an integer, and let $f \in L_p$ such that $\hat{f} \geq 0$. 

\footnote{This follows from the Orlicz-Pettis Theorem [2, p. 60].}
Then

1. \( f \) has an unconditionally convergent Fourier series in the \( L_p \) norm;
2. \( \| f \|_p = \| f \|_{D_p} = \| f \|_{S_p} \).

**Proof.** We claim that it is sufficient to show that
\[
\sup\{ \| f \ast g \|_p \mid g \text{ a trigonometric polynomial}, \| g \|_\infty \leq 1 \} \leq \| f \|_p.
\]
Since the trigonometric polynomials are dense in \( L_p \), we then deduce that
\[
\| f \ast h \|_p \leq \| f \|_p \| h \|_\infty, \quad h \in L_p.
\]
Thus \( f \in D_p = S_p \), so (1) follows. Now the above inequality gives that
\[
\| f \|_{D_p} \leq \| f \|_p.
\]
Since \( \| f \|_p \leq \| f \|_{S_p} \leq \| f \|_{D_p} \), (2) follows as well.

So, let \( g = \sum_{i=1}^n a_i \gamma_i \), where \( |a_i| \leq 1 \), and let
\[
h = f \ast g = \sum_{i=1}^n f(\gamma_i) a_i \gamma_i.
\]
Letting \( "[k]" \) denote \( k \)th convolution power, we have that
\[
\| h \|_p^2 = \int |h|^k \leq \| f \|_p \| h \|_2^2 = \| f \|_p^2 \| h \|_2^2 \leq \| f \|_p^2 \| h \|_2^2 \leq \| f \|_p^2 \| h \|_2^2.
\]
Since \( \| h \| \leq f \), we have that
\[
\| h \|_p^2 \leq \| f \|_p^2 \| h \|_2^2 = \| f \|_p^2 \| h \|_2^2 = \int |f|^{2k} = \| f \|_p^2.
\]
Therefore \( \| h \|_p \leq \| f \|_p \), and the theorem is established.

As a corollary, we have the following characterization of \( S_p \):

**Theorem 4.** Let \( p \) be an even integer, and let \( f \in L_p \). Then the following statements are equivalent:

1. \( f \) has an unconditionally convergent Fourier series in the \( L_p \) norm.
2. \( |f| \) is the Fourier transform of an \( L_p \) function.

**Proof.** Lemma 1 gives that (1) implies (2). If (2) holds and \( g = |f| \), then \( g \in S_p \) by the above theorem; so \( f \in S_p \) by Lemma 1.

**Remarks.** (1) For \( G \) a compact abelian group, Bochner’s theorem [9, p. 19] may be stated as follows:

If \( \phi \) is continuous on \( G \) and \( \phi \geq 0 \), then \( \phi \in S_\infty \). Thus Theorem 3 is an \( L_p \) analog of Bochner’s theorem, for \( p \) an even integer.

(2) The essential ingredient in the proof of Theorem 3 is the fact that, for \( p \) an even integer, if one multiplies the coefficients of a trigonometric polynomial by complex numbers of absolute value one, the largest \( L_p \) norm is obtained when the coefficients become non-negative. This was first observed by Hardy and Littlewood [6, p.
It is an open question as to what the maximum is for other values of \( p \).

For \( E \subseteq \Gamma, 1 < p < \infty \), let \( L_p^E \) be the set of functions in \( L_p(G) \) whose transforms vanish outside of \( E \). The set \( E \) is said to be of type \( \Lambda(p) \) if there exists \( q < p \) such that \( L_p^E = L_q^E \). For a discussion of equivalent notions, the reader is referred to [3].

The following lemma is known.

**Lemma 2.** Let \( 2 < p < \infty \). Then \( E \) is of type \( \Lambda(p) \) if and only if \( L_p^E \subseteq S_p \).

**Proof.** If \( E \) is of type \( \Lambda(p) \), then \( L_p^E = L_2^E \), so \( L_p^E \subseteq S_p \).

Suppose conversely that \( L_p^E \subseteq S_p \). Let \( f \in L_p^E \). Then (cf. [3, 14.3.2]) for some \( a : \Gamma \rightarrow \{-1, 1\} \), the function \( a \hat{f} \) is the transform of a function in \( L_p^E \). By Lemma 1, \( a(\hat{a}) = \hat{f} \) is also the transform of an \( L_p^E \) function. Therefore \( L_p^E = L_2^E \), so \( E \) is of type \( \Lambda(p) \).

As a direct consequence of the above lemma and Theorem 4, we have the following characterization of sets of type \( \Lambda(p) \):

**Theorem 5.** Let \( p > 2 \) be an even integer and let \( E \subseteq \Gamma \). Then \( E \) is of type \( \Lambda(p) \) if and only if \( |\hat{f}| \) is the transform of an \( L_p^E \) function whenever \( f \in L_p^E \).

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**References**


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