ELEMENTARY PROOF OF A THEOREM OF HELSON

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Abstract. An elementary proof is given of a theorem of Helson, to the effect that in a locally compact abelian group, the transform of a measure concentrated on a Helson set does not vanish at infinity.

$G$ is a locally compact abelian group whose dual is denoted $\Gamma$. A compact set $H$ in $G$ is called a Helson set if to every $f \in C(H)$ there corresponds an $f \in L^1(\Gamma)$ such that

$$F(x) = \int_{\Gamma} (\overline{\alpha}, \gamma) f(\gamma) d\gamma, \quad x \in H.$$

A simple characterization of Helson sets is the following (see e.g. [3, p. 115]): A compact set $H$ is a Helson set if and only if there exists a constant $B$ such that if $\sigma \in M(H)$, i.e. if $\sigma$ is a measure concentrated on $H$, then

(1) $||\sigma|| \leq B||\delta||_{\infty}$.

Helson's theorem (see a proof in [2] for $G = \mathbb{R}$, and a more difficult proof in [3, p. 119], for the general case) is the following:

Suppose $H$ is a Helson set, $\sigma \in M(H)$ and $\sigma \neq 0$. Then $\delta \in C_0(\Gamma)$, that is $\delta$ does vanish at infinity.

An interesting consequence of this theorem is that every Helson set has Haar measure zero.

We shall use the following well known facts:

If $\delta \in C_0(\Gamma)$ then $\sigma$ is a continuous measure.

If $\delta \in C_0(\Gamma)$ and if $g$ is any bounded Borel function, then $\langle g\delta \rangle^* \in C_0(\Gamma)$. In particular if $\sigma \in M(H)$, $\delta \in C_0(\Gamma)$, then multiplying $\sigma$ by a Borel function of modulus 1 we get a real measure on $H$ whose transform still vanishes at infinity.

If $\sigma$ is a real continuous measure on a set $S$, $\sigma(S) \neq 0$, and if $N$ is any positive integer, then we can partition $S$ into $N$ disjoint sets $S_i$ such that $\sigma(S_i) = N^{-1}\sigma(S)$ (see e.g. [1, p. 63, Exercise 9]).
We shall show in an elementary way, that for a certain positive integer $N$, depending on the Helson set $H$:

If $\sigma \in M(H)$, $\sigma$ real, $\sigma \neq 0$, $\delta \in C_0(\Gamma)$, then $\exists \mu \in M(H)$:

$$
\mu \text{ real, } \|\mu\| = \|\sigma\|, \quad \mu \in C_0(\Gamma), \quad \|\mu\|_\infty \leq (1 - 3/N)\|\delta\|_\infty.
$$

(2)

This will contradict (1), since by repeated application it yields a sequence $\mu_n$ in $M(H)$, with $\|\mu_n\| = \|\sigma\|, \|\mu\|_\infty \to 0$ and Helson's theorem will be proved.

Proof of (2). Choose the positive integer $N$ such that $2N^{-1}B \leq (1 - 2N^{-1})$. Let $\epsilon > 0$ be arbitrary. Let $K$ be a compact set in $\Gamma$ such that $|\delta(\gamma)| < \epsilon$ for $\gamma \in K$. We can decompose $H$ into a union of disjoint sets $E_i$, $i = 1, \ldots, n$, such that, for each $i$, $(x, \gamma)$ varies very little for $x \in E_i, \gamma \in K$. In particular, for $x_i \in E_i$,

$$
\left| \sum_i \sigma(E_i)(x_i, \gamma) - \delta(\gamma) \right| < \epsilon \quad \text{for } \gamma \in K.
$$

(3)

Each $E_i$ contains a set $E'_i$ such that $\sigma(E'_i) = N^{-1}\sigma(E_i)$ and we can choose $E'_i$ such that

$$
|\sigma|(E'_i) \leq N^{-1}|\sigma|(E_i).
$$

(4)

Let $\chi'$ be the characteristic function of $H' = \bigcup E'_i$ and put $\sigma' = \chi'\sigma$. By our choice of the $E_i$ we may suppose that, for $x_i \in E_i$,

$$
\left| \sum_i \sigma'(E'_i)(x_i, \gamma) - \delta'\gamma \right| < \epsilon \quad \text{for } \gamma \in K.
$$

(3')

Put $\mu = (1 - 2\chi')\sigma$. Since $|1 - 2\chi'| = 1$ we have $\|\mu\| = \|\sigma\|$. Also $\mu \in C_0(\Gamma)$.

Now for $\gamma \in K$, we have, by (3) and (3')

$$
|\mu(\gamma)| = |\delta(\gamma) - 2\delta'(\gamma)| \leq 3\epsilon + \left(1 - \frac{2}{N}\right)\|\delta(\gamma)(x_i, \gamma)\|_\infty
$$

\leq 3\epsilon + \left(1 - \frac{2}{N}\right)(\|\delta\| + \epsilon) \leq \left(1 - \frac{3}{N}\right)\|\delta\|_\infty \quad (\epsilon \text{ small})
$$

while for $\gamma \in K$, by (4),

$$
|\mu(\gamma)| < \epsilon + 2\|\sigma'\| \leq \epsilon + \frac{2}{N}\|\sigma\| \leq \epsilon + \frac{2}{N}B\|\delta\|_\infty
$$

\leq \epsilon + \left(1 - \frac{2}{N}\right)\|\delta\|_\infty \leq \left(1 - \frac{3}{N}\right)\|\delta\|_\infty.
$$

The proof of (2) is now complete.
Bibliography


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