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FACTORIZATION OF DIFFERENTIAL OPERATORS

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ABSTRACT. A necessary and sufficient condition for a differential operator of order n to be factorable into a product of operators of orders $n-r$ and r, for any $0<r<n$, is given.

Consider differential operators of the form

$$(*) \qquad Ly = s_0 y^{(n)} + s_1 y^{(n-1)} + \cdots + s_n y$$

with s_i continuous for $i=0, \cdots, n$ and $s_0(t) \neq 0$.

THEOREM. *Suppose L is an operator of order n of type (*) and $0<r<n$. Then there exist operators P and Q both of type (*) of orders $n-r$ and r respectively such that $L=PQ$ on some interval I if and only if there exist r linearly independent solutions $y_1, \cdots, y_r$ of $Ly=0$ satisfying the condition that the Wronskian $W_r = W_r(y_1 \cdots y_r) \neq 0$ on I.*

PROOF. Suppose $L=PQ$ where Q has order r. Since any solution of $Qy=0$ is also a solution of $Ly=0$, we can choose any r linearly independent solutions of $Qy=0$ and condition $W_r \neq 0$ will be satisfied.

On the other hand, assume $Ly_i=0$ for $i=1, \cdots, r$ and $W_r(y_1 \cdots y_r) \neq 0$. Let

$$Qy = \det \begin{bmatrix} y_1 \cdots y_r & y \\ y_1' \cdots y_r' & y' \\ y_1^{r} \cdots y_r^{r} & y^{r} \end{bmatrix} = q_0 y^{(r)} + q_1 y^{(r-1)} + \cdots + q_r y.$$

Note that $q_0 = W_r(y_1 \cdots y_r) \neq 0$ and $q_i \in C^{n+1-r-i}$. A direct computation shows that the coefficients p_i for $i=0, \cdots, n-r$ can be chosen successively so that the coefficients of $y^{(n)}, y^{(n-1)}, \cdots, y^{(n-r)}$ in the product PQy where $Py = p_0 y^{n-r} + \cdots + p_{n-r} y$ are $s_0, s_1, \cdots, s_r$, respectively. Hence the operator $N=L-PQ$ is of order less than r. But $y_1, \cdots, y_r$ are linearly independent solutions of $Ny=0$. Hence $N=0$ and $L=PQ$.

Although the above theorem may have been known to Ince—see

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the discussion on pp. 119 and 126–127 in [1]—the author has not seen an explicit statement of it anywhere in the literature.

The conditions $W_r \neq 0$ for $r = 1, \cdots, n-1$ are known to be necessary and sufficient for the factorability of an operator of type (*) into a "product" of first order operators—see [2].

References

1. E. L. Ince, *Ordinary differential equations*, Dover, New York, 1944. MR **6**, 65.

2. G. Pólya, *On the mean value theorem corresponding to a given linear homogeneous differential equation*, Trans. Amer. Math. Soc. **24** (1922), 312–324.

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