SHORTER NOTES

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ON THE HYPERSPACE OF SUBCONTINUA OF AN ARC-LIKE CONTINUUM

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Abstract. It is shown that the hyperspace of each arc-like continuum can be embedded in $E^3$.

W. R. R. Transue [1] in a beautiful note gave a positive answer to A. Connor's [2, p. 152] question "Can the hyperspace of subcontinua of the pseudoarc (with the Hausdorff metric) be embedded in $E^3". This note extends this result to arc-like continua, i.e. inverse limits on arcs originally called snake-like continua by R. H. Bing [3].

Here $\{W, f_i\}$ will denote the inverse limit system with indexing set the nonnegative integers and with each factor space $W$. The associated inverse limit space will be denoted by $\lim \{W, f_i\}$. See [4, p. 87] for a discussion of inverse limits. The hyperspace of continua of a space $X$, denoted by $C(X)$, is studied in [5]. The closed interval $[0, 1]$ will be called $I$.

Theorem. The hyperspace of continua of the inverse limit space $X = \lim \{I, f_i\}$ embeds in $E^3$.

Proof. There is no loss to assume that none of the maps $f_i$ is constant on an open set. Since a continuum in $I$ is either a closed interval or a point, $C(I)$ will be identified with

$$D = \{(x, y, z) \in E^3 | 0 \leq x \leq y \leq 1, z = 0\}.$$ 

Take $F_i: D \to D$ by

$$F_i(x, y, 0) = (\min f_i(t), \max f_i(t), 0), \quad t \in [x, y].$$

Now $F_i$ is the natural map from $C(I)$ to $C(I)$ induced by $f_i$.

J. Segal [6] proved that the hyperspace of continua of the inverse limit space $X$ is homeomorphic to $\lim \{C(I), F_i\}$. The proof of the theorem will be completed by embedding $\lim \{D, F_i\}$ in $E^3$. Now if each of the maps $F_i$ could be approximated by embeddings in $E^3$ in the

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sense of McCord [7, Theorem 2] then \( \lim \{ D, F_i \} \) embeds in \( E^3 \) by McCord's Theorem. To construct these approximations one must prove the following:

**Lemma.** If \( \epsilon > 0 \), then there is a homeomorphism \( h_i \) of \( E^3 \) onto \( E^3 \) such that \( \| h_i \| D, F_i \| < \epsilon \).

**Proof.** For each \( (x, y, 0) \in D \) take

\[
G_i(x, y, 0) = F_i(x, y, 0) + \left( 0, 0, \frac{\epsilon}{4} \frac{x + y}{2} \right) \quad \text{(vector addition)}.
\]

As \( f_i \) is not locally constant, a check will show that each point inverse of \( G_i \) is a point or a closed interval which does not separate \( D \). Thus \( G_i(D) \) is a topological disk and consequently there is a homeomorphism \( H_i \) of \( D \) onto \( G_i(D) \) such that \( \| H_i, G_i \| < \epsilon/4 \) by Radó [8, Theorem 2.17]. Next \( G_i(D) \) is tamed by an approximation theorem of R. H. Bing [9] which constructs a homeomorphism \( J_i \) of \( D \) into \( E^3 \), \( \| J_i, H_i \| < \epsilon/4 \) and with \( J_i(D) \) a polyhedron. So there is a homeomorphism \( h_i \) of \( E^3 \) onto \( E^3 \) with \( h_i \| D = J_i \) since \( J_i(D) \) is tame.

**Remark.** With more care one can construct this embedding so that the intersection of the embedded \( C(X) \) and the plane \( x = y \) is exactly the set of degenerate subcontinua of \( X \).

**References**


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