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CONDITIONAL EXPECTATIONS AND AN ISOMORPHIC CHARACTERIZATION OF L_1-SPACES

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ABSTRACT. Conditional expectations can be defined in Banach spaces whose elements can be represented as measurable functions. In the present paper it is shown that such a space (precisely a cyclic space) is isomorphic to an L_1-space if and only if the conditional expectations act as bounded operators for sufficiently many representations.

Let (Ω, Σ, μ) be a finite measure space and Σ_0 a subring of Σ with maximal element Ω_0; then for each Σ-measurable function f which is bounded on Ω one can consider the measure $\mu_0(\sigma) = \int_{\sigma\cap\Omega_0} f(\omega)\mu(d\omega)$; $\sigma\cap\Omega_0 \in \Sigma_0$. Since μ_0 is evidently absolutely continuous with respect to the restriction of μ to the subfield generated by Σ_0 and Ω, due to the Radon-Nikodym theorem, there exists a Σ_0-measurable function denoted $E(\Sigma_0, \mu)f$ for which

$$\int_\sigma f(\omega)\mu(d\omega) = \int_\sigma E(\Sigma_0, \mu)f\mu(d\omega); \qquad \sigma \in \Sigma_0.$$

Obviously the operator $E(\Sigma_0, \mu): f \to E(\Sigma_0, f)f$ can be extended uniquely to a contractive projection in $L_p(\Omega, \Sigma, \mu)$; $1 \leqq p \leqq +\infty$, which is called the conditional expectation relative to Σ_0.

However, if the L_p-norm is replaced by a general monotonic norm ρ in the sense of the theory of Banach function spaces (see for instance W. A. J. Luxemburg and A. C. Zaanen [9, Note I]), usually, $E(\Sigma_0, \mu)$ does not act as a bounded operator in L_ρ—the space of all Σ-measurable functions for which $\rho(f) < +\infty$, even when we assume that $L_1(\Omega, \Sigma, \mu) \supset L_\rho \supset L_\infty(\Omega, \Sigma, \mu)$. Furthermore, a Banach function space L_ρ admits many isometric representations; e.g. to every positive function $\phi \in L_\rho$ whose support is Ω one can define a new norm $\rho_\phi(f) = \rho(\phi f)$, obtaining in this way a new Banach function space L_{ρ_ϕ} which is isometric to L_ρ and satisfies $L_1(\Omega, \Sigma, \phi\mu) \supset L_{\rho_\phi} \supset L_\infty(\Omega, \Sigma, \phi\mu)$.

The main result of this paper states that L_ρ is isomorphic to an L_1-space over a finite measure space provided for every subring Σ_0 of

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Σ and positive function $\phi \in L_p$ whose support is Ω the conditional expectation $E(\Sigma_0, \phi\mu)$ is a *bounded* operator in L_{p_ϕ}.

Since a Banach space isomorphic to an L_1-space is not in general a Banach function space, in order to have a complete characterization we will consider the more natural context (for this purpose) of cyclic spaces introduced by W. G. Bade [3] rather than that of Banach function spaces (for other characterizations using cyclic spaces see [11]).

Conditional expectations in cyclic spaces. We shall start by summarizing some notions and results needed in the sequel. A Boolean algebra of projections $\mathfrak{B}$ in a Banach space X is called σ-complete (cf. W. G. Bade [2]) provided for every sequence $P_n \in \mathfrak{B}$; $n=1, 2, \cdots$, the projections $\bigvee_{n=1}^{\infty} P_n$ and $\bigwedge_{n=1}^{\infty} P_n$ exist in $\mathfrak{B}$ and satisfy

$$\left(\bigvee_{n=1}^{\infty} P_n\right) X = \operatorname{clm}_n \{P_n X\}; \qquad \left(\bigwedge_{n=1}^{\infty} P_n\right) X = \bigcap_{n=1}^{\infty} \{P_n X\}.$$

It is well known that $\mathfrak{B}$ can be regarded as a spectral measure $P(\cdot)$ defined on the Borel sets Σ of its Stone space Ω and it follows from W. G. Bade [2, Theorem 2.2] and N. Dunford [4] that there exists a constant K such that for every Borel bounded function f, the integral $S(f) = \int_\Omega f(\omega) P(d\omega)$ exists in the uniform operator topology and satisfies the inequality: $\|S(f)\| \leq K \sup_{\omega \in \Omega} |f(\omega)|$. For f unbounded we can consider $S(f)$ as an unbounded operator whose domain is

$$D(S(f)) = \left\{ x \,\middle|\, x \in X, \lim_{m \to \infty} \int_{e_m} f(\omega) P(d\omega) x \text{ exists} \right\}$$

and $e_m = \{\omega \mid \omega \in \Omega, |f(\omega)| \leq m\}$; $m = 1, 2, \cdots$. According to W. G. Bade [3], X is a *cyclic space* relative to a σ-complete Boolean algebra of projections $\mathfrak{B}$ if there is $x_0 \in X$ such that $X = \mathfrak{M}(x_0) = \operatorname{clm}\{P x_0 \mid P \in \mathfrak{B}\}$. In this case, by W. G. Bade [3, Theorem 4.5], $X = \mathfrak{M}(x_0) = \{S(f) x_0 \mid x_0 \in D(S(f))\}$. Let us also mention that for every cyclic space $X = \mathfrak{M}(x_0)$ there exists a functional $x_0^* \in X^*$, which will be called Bade functional, with the following properties:

(i) $x_0^* P x_0 \geq 0; P \in \mathfrak{B}$;

(ii) if $x_0^* P x_0 = 0$ for some $P \in \mathfrak{B}$, then $P = 0$ (cf. W. G. Bade [2, Theorem 3.1]).

With this preparation we can state our principal result which is contained in the following theorem.

THEOREM 1. *A Banach space X is isomorphic to an L_1-space over a finite measure space if and only if:*

(a) *X is a cyclic space $\mathfrak{M}(x_0)$; $x_0 \in X$, relative to some σ-complete Boolean algebra of projections $\mathfrak{B}$, and*

(b) *there exists a Bade functional $x_0^* \in X^*$ such that the series*

$$\sum_{n=1}^{\infty} \frac{x_0^* P(\sigma_n) S(f) x_0}{x_0^* P(\sigma_n) S(\phi) x_0} P(\sigma_n) S(\phi) x_0;$$

$$S(f)x_0, S(\phi)x_0 \in \mathfrak{M}(x_0); \quad \phi(\omega) > 0; \quad \omega \in \Omega,$$

converges strongly in X for every sequence of disjoint sets $\sigma_n \in \Sigma$; $n = 1, 2, \cdots$.

PROOF. Let τ be an isomorphism between an L_1-space $L_1(T, \mathfrak{Z}, m)$; $m(T) < +\infty$ and X; $x_0 = \tau(1)$; $x_0^* = (\tau^*)^{-1}(1)$ and $\mathfrak{F} = \{F(e) \mid e \in T; F(e)\tau(f) = \tau(\chi_e f); f \in L_1(T, \mathfrak{Z}, m)\}$. Obviously X is a cyclic space relative to the σ-complete Boolean algebra of projections $\mathfrak{F}$ for which the series in condition (b) of the theorem converges and its sum is bounded in norm by $\|\tau\| \cdot \|\tau^{-1}\| \cdot \|S(f)x_0\|$.

Conversely, let us set

$$Q(S(f)x_0) = \sum_{n=1}^{\infty} \frac{x_0^* P(\sigma_n) S(f) x_0}{x_0^* P(\sigma_n) S(\phi) x_0} P(\sigma_n) S(\phi) x_0; \qquad S(f)x_0 \in X.$$

One can easily see that Q is a linear projection in X. In order to show that Q is bounded assume there exist $S(f_n)x_0 \in X$; $\|S(f_n)x_0\| = 1$; $n = 1, 2, \cdots$, such that $\|Q(S(f_n)x_0)\| \geqq n^3$. The inequality

$$\|S(|f_n|)x_0\| = \left\| S\left(\frac{|f_n|}{f_n}\right) S(f_n)x_0 \right\| \leqq K \|S(f_n)x_0\|$$

shows that $S(|f_n|)x_0 \in X$ and $\|S(|f_n|)x_0\| \leqq K$; $n = 1, 2, \cdots$. Set $Q(S(f_n)x_0) = S(g_n)x_0$; $Q(S(|f_n|)x_0) = S(h_n)x_0$; since $|g_n(\omega)| \leqq h_n(\omega)$; $\omega \in \Omega$, we obtain

$$\|S(h_n)x_0\| \geqq \frac{1}{K} \|S(|g_n|)x_0\| \geqq \frac{1}{K^2} \|S(g_n)x_0\| \geqq \frac{n^3}{K^2}.$$

Thus for $S(f)x_0 = \sum_{n=1}^{\infty} (S(|f_n|)x_0/n^2) \in X$ we have

$$\|Q(S(f)x_0)\| \geqq \frac{1}{K} \left\| Q\left(\frac{S(|f_n|)x_0}{n^2}\right) \right\| = \frac{1}{Kn^2} \|S(h_n)x_0\| \geqq \frac{n}{K^3}$$

which shows that Q is not defined in $S(f)x_0 \in X$ i.e. condition (b) of the theorem does not hold. We have to point out that Q depends on the choice of $S(\phi)x_0 \in X$ and $\sigma_n \in \Sigma$; $n = 1, 2, \cdots$.

Now, let $\{\delta_n'\}$ and $\{\delta_n''\}$ be two sequences of mutually disjoint sets; $\delta_n', \delta_n'' \in \Sigma$; $n=1, 2, \cdots$, $(\bigcup_{n=1}^{\infty}\delta_n') \cap (\bigcup_{n=1}^{\infty}\delta_n'') = \varnothing$ for which $P(\delta_n')$ as well as $P(\delta_n'')$ are nonzero projections. Set $\mu(\sigma) = x_0^* P(\sigma) x_0$; $\sigma \in \Sigma$; $a_n = \min\{\mu(\delta_n'), \mu(\delta_n'')\}$ and

$$S(\phi)x_0 = \sum_{n=1}^{\infty} \frac{a_n}{2^n} \frac{P(\delta_n')x_0}{\mu(\delta_n')} + \sum_{n=1}^{\infty} \frac{a_n}{2^n} \frac{P(\delta_n'')x_0}{\mu(\delta_n'')} + P\left(\Omega - \left(\bigcup_{n=1}^{\infty} \delta_n' \cup \bigcup_{n=1}^{\infty} \delta_n''\right)\right)x_0.$$

Since $\phi(\omega) > 0$; $\omega \in \Omega$, we can consider the projection Q corresponding to $S(\phi)x_0$ and the partition $\sigma_n = \delta_n' \cup \delta_n''$; $n = 1, 2, \cdots$. Then

$$Q(P(\delta_n')x_0) = \frac{2^{n-1}}{a_n} \mu(\delta_n') \left[\frac{a_n}{2^n} \frac{P(\delta_n')x_0}{\mu(\delta_n')} + \frac{a_n}{2^n} \frac{P(\delta_n'')x_0}{\mu(\delta_n'')}\right]$$

and further

$$Q\left(\frac{P(\delta_n')x_0}{\mu(\delta_n')}\right) = \left(\frac{P(\delta_n')x_0}{\mu(\delta_n')} + \frac{P(\delta_n'')x_0}{\mu(\delta_n'')}\right) \Big/ 2$$

and similarly:

$$Q\left(\frac{P(\delta_n'')x_0}{\mu(\delta_n'')}\right) = \left(\frac{P(\delta_n')x_0}{\mu(\delta_n')} + \frac{P(\delta_n'')x_0}{\mu(\delta_n'')}\right) \Big/ 2.$$

This implies that a series $\sum_{n=1}^{\infty} c_n P(\delta_n')x_0/\mu(\delta_n')$ converges if and only if the series $\sum_{n=1}^{\infty} c_n P(\delta_n'')x_0/\mu(\delta_n'')$ does, i.e., the basis $\{P(\delta_n')x_0/\mu(\delta_n')\}$ is equivalent to the basis $\{P(\delta_n'')x_0/\mu(\delta_n'')\}$. If π is a permutation of the natural numbers, the bases $\{P(\delta_n')x_0/\mu(\delta_n')\}$ and $\{P(\delta_{\pi(n)}')x_0/\mu(\delta_{\pi(n)}')\}$ will be equivalent since both are equivalent to $\{P(\delta_n'')x_0/\mu(\delta_n'')\}$. In the terminology of I. Singer [**10**] (see also M. J. Kadec and A. Pełczyński [**7**]) this means that both bases $\{P(\delta_n')x_0/\mu(\delta_n')\}$ and $\{P(\delta_n'')x_0/\mu(\delta_n'')\}$ are symmetric and thus by [**7**, Theorem 5] there exists M' and M'' such that

$$\frac{1}{\|x_0^*\|} \leqq \frac{\|P(\delta_n')x_0\|}{\mu(\delta_n')} \leqq M'; \quad \frac{1}{\|x_0^*\|} \leqq \frac{\|P(\delta_n'')x_0\|}{\mu(\delta_n'')} \leqq M''; \qquad n = 1, 2, \cdots.$$

Consequently, for any partition $\{\delta_n\}$ there exists M such that $\|P(\delta_n)x_0\|/\mu(\delta_n) \leqq M$; $n = 1, 2, \cdots$, (since we can take $\delta_n' = \delta_{2n-1}$ and $\delta_n'' = \delta_{2n}$; $n = 1, 2, \cdots$).

The next step will be to show that for any partition $\{\delta_n\}$ the basis $\{P(\delta_n)x_0/\mu(\delta_n)\}$ is equivalent to the natural basis of l_1. Indeed, if a series $\sum_{n=1}^{\infty} c_n P(\delta_n)x_0/\mu(\delta_n)$ is convergent, then it is easy to see that the series $\sum_{n=1}^{\infty}|c_n|P(\delta_n)x_0/\mu(\delta_n)$ is convergent too. By repeating the previous part of the proof for

$$S(\psi)x_0 = \sum_{n=1}^{\infty} |c_n| \frac{P(\delta_n)x_0}{\mu(\delta_n)} + P\left(\Omega - \bigcup_{n=1}^{\infty} \delta_n\right)x_0$$

instead of x_0 (which is possible since $\psi(\omega)>0$; $\omega\in\Omega$) we get $\sum_{n=1}^{\infty}\|P(\delta_n)S(\psi)x_0\| < +\infty$ which implies the convergence of $\sum_{n=1}^{\infty}|c_n|$. Since the converse is obvious the assertion is completely proved.

The crucial point in the proof is to show that

$$\sup_{0\neq\delta\in\Sigma} \frac{\|P(\delta)x_0\|}{\mu(\delta)} < +\infty.$$

Suppose there exists a sequence $\eta_n\in\Sigma$, for which $\|P(\eta_n)x_0\|/\mu(\eta_n) \geqq n$; $n=1, 2, \cdots$, and $\mu(\eta_n)\neq 0$. We shall construct by induction another sequence $\{\sigma_n\}$ with the properties

(i) $\sigma_n \cap \sigma_i$ is equal to σ_n or $\emptyset$;

(ii) $\|P(\sigma_n)x_0\|/\mu(\sigma_n) \geqq n$; $1\leqq i \leqq n-1$.

Indeed, set $\sigma_1=\eta_1$ and assume that $\sigma_1, \cdots, \sigma_n$ are already constructed and satisfying the conditions (i) and (ii). The following equality

$$\eta_{n+1} = \left(\eta_{n+1} - \bigcup_{k=1}^{n} \sigma_k\right) \cup (\eta_{n+1}\cap\sigma_n) \cup \bigcup_{k=1}^{n-1}\left(\eta_{n+1}\cap\left(\sigma_k - \bigcup_{j=k+1}^{n}\sigma_j\right)\right)$$

splits η_{n+1} into $n+1$ disjoint sets; hence

$$\begin{aligned}
&\left\|P\left(\eta_{n+1} - \bigcup_{n=1}^{n} \sigma_k\right)x_0\right\| + \|P(\eta_{n+1}\cap\sigma_n)x_0\| \\
&\qquad + \sum_{k=1}^{n-1}\left\|P\left(\eta_{n+1}\cap\left(\sigma_k - \bigcup_{j=k+1}^{n}\sigma_j\right)\right)x_0\right\| \\
&\geqq \|P(\eta_{n+1})x_0\| \geqq (n+1)\mu(\eta_{n+1}) \\
&= (n+1)\left[\mu\left(\eta_{n+1} - \bigcup_{k=1}^{n}\sigma_k\right) + \mu(\eta_{n+1}\cap\sigma_n)\right. \\
&\qquad \left. + \sum_{k=1}^{n-1}\mu\left(\eta_{n+1}\cap\left(\sigma_k - \bigcup_{j=k+1}^{n}\sigma_j\right)\right)\right].
\end{aligned}$$

Thus, at least for one of these disjoint sets, which will be denoted η_{n+1} we have $\|P(\eta_{n+1})x_0\| \geqq (n+1)\mu(\sigma_{n+1})$ and in this way all conditions imposed on $\{\sigma_n\}$ will hold.

Condition (i) satisfied by the sequence $\{\sigma_n\}$ shows that it contains either a subsequence of disjoint sets or a nested subsequence $\{\sigma_{n_j}\}$. The first possibility leads immediately to a contradiction of condition (ii) while in the second case we will have the basis

$$\{P(\sigma_{n_j} - \sigma_{n_{j+1}})x_0/\mu(\sigma_{n_j} - \sigma_{n_{j+1}})\}$$

which is equivalent to the natural basis of l_1. In view of the closed graph theorem this implies the existence of a constant A such that:

$$\left\| \sum_{j=1}^{\infty} \alpha_j \frac{P(\sigma_{n_j} - \sigma_{n_{j+1}})x_0}{\mu(\sigma_{n_j} - \sigma_{n_{j+1}})} \right\| \leqq A \sum_{j=1}^{\infty} |\alpha_j|$$

for any sequence $(\alpha_j) \in l_1$. Taking $\alpha_j = \mu(\sigma_{n_j} - \sigma_{n_{j+1}})/\mu(\sigma_{n_k} - \sigma_{n_{m+1}})$; $k \leqq j \leqq m$, we obtain

$$\left\| \sum_{j=k}^{m} \frac{P(\sigma_{n_j} - \sigma_{n_{j+1}})x_0}{\mu(\sigma_{n_k} - \sigma_{n_{m+1}})} \right\| \leqq A$$

i.e.

$$\left\| \frac{P(\sigma_{n_k})x_0 - P(\sigma_{n_{m+1}})x_0}{\mu(\sigma_{n_k}) - \mu(\sigma_{n_{m+1}})} \right\| \leqq A.$$

But $\mu(\sigma_{n_{m+1}}) \leqq K\|x_0\|/n_{m+1}$ i.e. $\lim_{m\to\infty} \mu(\sigma_{n_{m+1}}) = 0$ and therefore $\lim_{m\to\infty} \|P(\sigma_{n_{m+1}})x_0\| = 0$. Thus $\|P(\sigma_{n_k})\|/\mu(\sigma_{n_k}) \leqq A$; $k = 1, 2, \cdots$, which contradicts again condition (ii).

In conclusion we have proved the existence of a constant L such that

$$\|P(\delta)x_0\|/\mu(\delta) \leqq L; \qquad \delta \in \Sigma;\ \mu(\delta) \neq 0.$$

Finally, let f be a simple function; one can easily see that

$$\frac{1}{K\|x_0^*\|} \int_\Omega |f(\omega)|\, \mu(d\omega) \leqq \frac{1}{K} \|S(|f|)x_0\| \leqq \|S(f)x_0\|$$
$$\leqq L \int_\Omega |f(\omega)|\, \mu(d\omega)$$

which shows that $X = \mathfrak{M}(x_0)$ is isomorphic to $L_1(\Omega, \Sigma, \mu)$.

Let $X = \mathfrak{M}(x_0)$ be a cyclic space relative to a σ-complete Boolean algebra of projections $\mathfrak{B}$ (regarded as a spectral measure $P(\cdot)$ on (Ω, Σ) and x_0^* a Bade functional. It is quite clear that for any $x_\phi = S(\phi)x_0 \in X$; $\phi(\omega) > 0$; $\omega \in \Omega$, we have $\mathfrak{M}(x_0) = \mathfrak{M}(x_\phi)$. Considering the positive measure $\nu_\phi(\sigma) = x_0^* P(\sigma)x_\phi$ we can define the conditional expectation $E(\Sigma_0, \nu_\phi)$ relative to a subring Σ_0 of Σ as the operator in $\mathfrak{M}(x_\phi)$

$=X$ which assigns to $S(f)x_\phi \in \mathfrak{M}(x_\phi)$ the vector $E(\Sigma_0, \nu_\phi)S(f)x_\phi = S(h)x_\phi$ where h would be the Radon-Nikodym derivative of the measure $x_0^*P(\sigma)S(f)x_\phi$; $\sigma \in \Sigma_0$, with respect to the restriction of ν_ϕ to Σ_0 ($S(f)x_\phi$ belongs to the domain of $E(\Sigma_0, \nu_\phi)$ if and only if $x_\phi \in D(S(h))$).

Now the previous theorem can be restated as follows:

THEOREM 2. *A Banach space X is isomorphic to an L_1-space over a finite measure space if and only if*:

(a) *X is a cyclic space $\mathfrak{M}(x_0)$; $x_0 \in X$, relative to some σ-complete Boolean algebra of projections $\mathfrak{B}$, and*

(b) *there exists a Bade functional $x_0^* \in X^*$ such that for every subring Σ_0 of Σ and $x_\phi = S(\phi)x_0 \in X$; $\phi(\omega) > 0$; $\omega \in \Omega$, the conditional expectation $E(\Sigma_0, \nu_\phi)$ is a linear bounded projection in $X = \mathfrak{M}(x_\phi)$.*

PROOF. It suffices to observe that the series involved in the statement of Theorem 1 converges and its sum is $E(\Sigma_0, \nu_\phi)S(f\phi^{-1})x_\phi$ where Σ_0 is the subring generated by the sets σ_n; $n = 1, 2, \cdots$. Q.E.D.

REMARKS. 1. Using Banach function spaces instead of cyclic spaces might simplify the statement of Theorem 2 but only a sufficient condition for such spaces to be isomorphic to an L_1-space can be obtained. The precise assertion appears in the introduction.

2. In defining the measures ν_ϕ we use the same Bade functional x_0^*; if instead we set $\lambda_\phi(\cdot) = x_\phi^* P(\cdot)x_\phi$, where x_ϕ^* depends on x_ϕ, L_ρ will be isometric to an L_p-space; $1 \leqq p < +\infty$, provided all the conditional expectations $E(\Sigma_0, \lambda_\phi)$ will be contractive projections in L_ρ (cf. T. Ando [1]).

3. The problems discussed in this paper are related to the so called "leveling property" of a norm ρ in a Banach function space (cf. H. W. Ellis and I. Halperin [5]) and to the property (J) introduced by N. E. Gretsky [6]. It follows from Theorem 1 that unless a weakly sequentially complete Banach function space is isomorphic to an L_1-space, there exists always an isometric representation L_{ρ_ϕ} of L_ρ in which $\sup_\phi \rho_\phi(f) = +\infty$ (in the notation of [6]), and consequently ρ_ϕ does not admit an equivalent rearrangement-invariant norm with respect to the measure ν_ϕ (cf. W. A. J. Luxemburg [8, Theorem 14.4]).

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