OSCILLATION CRITERIA FOR ELLIPTIC SYSTEMS

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ABSTRACT. Oscillation criteria are established for quasilinear elliptic partial differential systems of second order in unbounded domains of Euclidean space. The main departures from earlier investigations are: (1) systems of partial differential equations are considered; (2) the equations are nonlinear; (3) the matrices involved are not required to be positive definite; and (4) a direct method of the Calculus of Variations is used rather than the earlier majorization method.

1. **Introduction.** The first general oscillation criterion for partial differential equations was obtained by I. M. Glazman in 1958 [5]. This is a natural extension of the classical Kneser-Hille criterion to the Schrödinger equation $\nabla^2 u + B(x)u = 0$ $(x \in R^n)$ and involves the limiting behavior of the minimum of $B(x)$ on the sphere $|x| = r$ as $r \to \infty$. An analogue of Glazman's theorem and other oscillation criteria were obtained recently for second order linear and quasilinear elliptic equations by Allegretto [1], [2], [3], Headley and Swanson [7], and Swanson [12]. Some extensions to elliptic equations of order $2m$ also have been derived by a modification of Glazman's majorization method [4], [6].

The oscillation criteria cited above all involve either limits at infinity or integrals of suitable majorants (or minorants) of the coefficients of the differential operator. The results are derived by comparison of the given differential equation with a separable equation possessing a solution with an oscillatory radial function as a factor, and use of any of the well-established one dimensional oscillation criteria.

Our first purpose here is to develop oscillation criteria for partial differential equations which do not depend upon one dimensional criteria: We dispense with the majorization procedure altogether and use a direct method of the type used in the Calculus of Variations. Unlike all earlier results, our criteria are not one dimensional in character if $n > 1$. Specialization to the case $n = 1$ yields either a new oscillation criterion for ordinary differential equations (or systems) or an independent proof of a known result. Our criteria involve integrals

of the actual coefficient functions appearing in the differential equation (rather than majorants) over the unbounded domain under consideration.

As a second departure from earlier investigations, we consider *systems* of partial differential equations. In the linear case, comparison theorems of Sturm's type for strongly elliptic systems were obtained by Kuks [10], Kreith [9], and Swanson [15]. Here we shall employ the matrix formulation of the system, and increase the generality by considering both the quasilinear case and inequalities for associated differential forms.

Oscillation criteria for matrix *ordinary* differential equations were obtained recently by Tomastik [16] and Swanson [14]. The oscillatory character of matrix solutions was established only for *prepared* matrices (also called *conjugate* matrices by some authors; in the linear case, the column vectors of a prepared matrix constitute a conjugate basis for the differential system in the language of M. Morse [11]). Indeed, easy examples show that the conclusions of the theorems in [14] and [16] are false without the hypothesis that the matrix solutions are prepared. Here we shall give a direct extension of the one dimensional results to n-dimensions without appealing to the former.

2. **Definitions and notations.** Oscillation criteria will be established for the matrix partial differential equation

$$(1) \quad LV \equiv \sum_{i,j=1}^{n} D_i[A_{ij}(x)D_jV] + B(x, V, D_1V, \cdots, D_nV)V = 0,$$

$$A_{ij} = A_{ji},$$

and more generally for the inequality $V^T L V \leqq 0$ (as a form) in unbounded domains G in n-dimensional Euclidean space R^n. As usual, points in R^n will be denoted by $x = (x^1, \cdots, x^n)$ and differentiation with respect to x^i by D_i, $i = 1, \cdots, n$. The following assumptions will be made throughout:

ASSUMPTIONS.

(1) B, V, and each A_{ij} are m-square real matrix functions.

(2) Each A_{ij} is symmetric and of class $C^1(G)$.

(3) The mn-square matrix $(A_{ij}(x))$, $i,j = 1, \cdots, n$, is positive semidefinite in G.

(4) B is symmetric, continuous, and bounded below (as a form) for $x \in G$ and for all values of the entries of V and its partial derivatives.

The domain $\mathfrak{D}$ of L is defined as the set of all m-square matrix functions $V \in C^2(G)$.

Let M be a nonempty regular bounded domain within G, and let F be the functional defined by

$$F[u, V; M] = \int_M \left[\sum_{i,j} D_i u^T A_{ij}(x) D_j u \right.$$

$$\left. - u^T B(x, V(x), D_1 V(x), \cdots, D_n V(x)) u \right] dx$$

for any piecewise C^1 m-vector function u on M and any $V \in \mathfrak{D}$.

A matrix $V \in C^1(G)$ is said to be *prepared* (or conjugate) iff $W_i(x)$ is symmetric for each $i = 1, \cdots, n$, where

$$(2) \qquad W_i(x) = \sum_{j=1}^{n} V^T(x) A_{ij}(x) D_j V(x).$$

This definition is analogous to M. Morse's definition [11] of a conjugate basis for an ordinary linear system ($n=1$), which has subsequently been modified and extended by Kuks [10], Tomastik [16], Swanson [14], and others.

The following natural extension of the well-known one dimensional definition of an oscillatory differential equation (or system) will be used in the sequel. The notation

$$G_a = G \cap \{x \in R^n : |x| > a\}, \qquad 0 < a < \infty,$$

will be used throughout.

DEFINITION. The matrix differential inequality $V^T L V \leq 0$ is termed *(weakly) oscillatory* in an unbounded domain $G \subset R^n$ iff every prepared solution V of the inequality has the property that $\det V(x)$ vanishes at some point of G_a for all $a > 0$.

This definition has been used by Kuks [10] in the linear case, by Kreith [8] when $m = 1$, and by Tomastik [16] when $n = 1$.

3. Oscillation criteria.

THEOREM 1. *The inequality $V^T L V \leq 0$ (as a form) is oscillatory in an unbounded domain G if*

(1) *for arbitrary $a > 0$, G_a contains a nonempty regular bounded domain M_a; and*

(2) *there exists a piecewise C^1 vector function u_a on $\overline{M}_a$ such that u_a vanishes identically on ∂M_a and $F[u_a, V; M_a] < 0$ for every differentiable matrix V with $\det V(x) \neq 0$ on $\overline{M}_a$.*

PROOF. Suppose to the contrary that $V^T L V \leq 0$ is not oscillatory, i.e. there exists a positive number a and a prepared solution V of the inequality such that $\det V(x) \neq 0$ in G_a. Then a unique solution $w(x)$ of $u(x) = V(x)w(x)$ exists in G_a for every m-vector $u(x)$. The following

identity, similar to that in $[13, \text{p. } 188]$ is easily verified by differentiation for any piecewise C^1 vector function u:

$$
\begin{aligned}
\sum_{i,j} (V D_i w)^T A_{ij}(x) V D_j w + \sum_i D_i \Big[u^T \sum_j A_{ij}(x) (D_j V) w \Big] \\
(3) \qquad = \sum_{i,j} D_i u^T A_{ij}(x) D_j u - u^T B(x, V, D_1 V, \cdots, D_n V) u \\
+ (Vw)^T (LV) w + \sum_i w^T [W_i(x) - W_i^T(x)] D_i w,
\end{aligned}
$$

where $W_i(x)$ is given by (2). Since V is prepared, the last term on the right side is identically zero. Taking u to be the vector function u_b $(b > a)$ in the hypothesis (2) and integrating over M_b, we obtain the contradiction $F[u_b, V; M_b] \geqq 0$.

The following result is an application of Theorem 1 in the case that G coincides with R^n.

THEOREM 2. *The inequality $V^T L V \leqq 0$ is oscillatory in R^n if (in addition to the assumptions stated in §2)*

    (1) *the mn-square matrix $(A_{ij}(x))$ is bounded above (as a form); and*

    (2) *there exists a diagonal element B_{kk} of B such that*

$$(4) \qquad \lim_{a \to \infty} a^{1-n} \int_{S_a} B_{kk}[x, V(x), D_1 V(x), \cdots, D_n V(x)] dx = + \infty$$

for every differentiable matrix $V(x)$ with $\det V(x) \neq 0$ for all sufficiently large $|x|$, where S_a denotes the ball $\{x \in R^n : |x| < a\}$.

PROOF. The domain M_a of Theorem 1 will be specialized to the annular domain

$$M_{ab} = \{x \in R^n : a < |x| < b\}$$

$(b > a+2)$. Define the piecewise C^1 vector function u_{ab} on $\overline{M}_{ab}$ by $u_{ab}(x) = e_k \phi_{ab}(x)$, where e_k is the unit vector with 1 in the kth position and 0 elsewhere, and

$$
\begin{aligned}
\phi_{ab}(x) &= r - a && \text{if } a \leqq r \leqq a+1, \\
&= 1 && \text{if } a+1 < r \leqq b-1, \\
&= b - r && \text{if } b-1 < r \leqq b
\end{aligned}
$$

$(r = |x|)$. Clearly M_{ab} is a regular bounded domain with $M_{ab} \subset G_a$, and $u_{ab} = 0$ on ∂M_{ab} for arbitrary a and b, $a > 0$, $b > a+2$. The conclusion will then follow from Theorem 1 if it can be shown that $F[u_{ab}, V; M_{ab}] < 0$ for sufficiently large b and for every differentiable V with $\det V(x) \neq 0$ on $\overline{M}_{ab}$.

To prove this, first observe that

$$\int_{M_{ab}} \sum_{i,j} D_i u_{ab}^T A_{ij}(x) D_j u_{ab}\, dx$$

(5)
$$\leq \alpha \int_{M_{ab}} \sum_i (D_i \phi_{ab})^2 dx$$

$$= \alpha \int_{M_{ab}} |\nabla \phi_{ab}|^2 dx$$

$$= \alpha n^{-1}\omega_n[(a+1)^n - a^n + b^n - (b-1)^n],$$

where α is an upper bound on $(A_{ij}(x))$ and ω_n denotes the area of the unit $(n-1)$-sphere. ($\omega_1 = 1$ by convention.) Similarly

$$\int_{M_{ab}} u_{ab}^T B(x, V, D_1 V, \cdots, D_n V) u_{ab} dx$$

(6)
$$\geq \beta \omega_n \int_a^{a+1} (r-a)^2 r^{n-1} dr$$

$$+ \int_{M_{a+1,b-1}} B_{kk}(x, V, D_1 V, \cdots, D_n V) dx$$

$$+ \beta \omega_n \int_{b-1}^b (b-r)^2 r^{n-1} dr,$$

where β is a lower bound on $B(x, V, D_1 V, \cdots, D_n V)$. Subtracting (6) from (5) we find that

$$b^{1-n} F[u_{ab}, V; M_{ab}] \leq b^{1-n}[g(a) + h(b)]$$

(7)
$$- b^{1-n} \int_{M_{a+1,b-1}} B_{kk}(x, V, D_1 V, \cdots, D_n V) dx,$$

where $g(a)$ is independent of b and $b^{1-n}h(b)$ is bounded in $2 \leq b < \infty$. However, the last term on the right side of (7) has limit $-\infty$ as $b \to \infty$ by the hypothesis (4). Hence $F[u_{ab}, V; M_{ab}] < 0$ for sufficiently large b, and the conclusion follows from Theorem 1.

Theorem 2 remains true if (4) is replaced by the obviously stronger condition

(8)
$$\lim_{a \to \infty} a^{1-n} \int_{S_a} \operatorname{tr} B[x, V(x), D_1 V(x), \cdots, D_n V(x)] dx = +\infty$$

where $\operatorname{tr} B$ denotes the trace of B. In the special case that B is positive definite, (4) can be replaced by the equivalent condition

$$(9) \quad \lim_{a \to \infty} a^{1-n} \int_{S_a} \lambda[x, V(x), D_1V(x), \cdots, D_nV(x)]dx = + \infty$$

where λ denotes the largest eigenvalue of B.

Theorem 2 requires neither (A_{ij}) nor B to be positive definite. The positive semidefiniteness of (A_{ij}) is essential, but an analogue of Theorem 2 can be obtained without the assumptions that B is bounded below and (A_{ij}) is bounded above. If $n = 1$, conditions (4), (8), and (9) reduce to those given recently by Tomastik [16] and Swanson [14].

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