CONTINUUM NEIGHBORHOODS AND FILTER BASES

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Abstract. In this paper we prove that if \( \Gamma \) is a filterbase of closed subsets of a compact Hausdorff space then \( T(\bigcap \Gamma) = \bigcap \{ T(G) \mid G \in \Gamma \} \), where \( T(A) \) denotes the set of those points for which every neighborhood which is a continuum intersects \( A \) nonvoidly.

Introduction. In this paper \( S \) denotes a compact Hausdorff space. If \( p \in S \) and \( W \subseteq S \), then \( W \) is a continuum neighborhood of \( p \) iff \( W \) is a subcontinuum of \( S \) and \( p \in \text{Int}(W) \). If \( A \subseteq S \), \( T(A) \) denotes the complement of the set of those points \( p \) of \( S \) for which there exists a continuum neighborhood which is disjoint from \( A \) \([1]\). \( S \) is said to be \( T\)-additive iff for every collection \( A \) of closed subsets of \( S \) whose union is closed, \( T(\bigcup A) = \bigcup \{ T(L) \mid L \in A \} \) \([2]\). The following three theorems are established.

Theorem A. Let \( \Gamma \) be a filterbase of closed subsets of \( S \). Then \( T(\bigcap \Gamma) = \bigcap \{ T(G) \mid G \in \Gamma \} \).

Theorem B. \( S \) is \( T\)-additive iff for each pair \( A, B \) of closed subsets of \( S \), \( T(A \cup B) = T(A) \cup T(B) \).

Theorem C. Let \( A \) be a closed subset of \( S \). If \( K \) is a component of \( T(A) \) then \( T(A \cap K) = K \cup T(\emptyset) \).

Theorem A is used in establishing Theorems B and C. Theorem C is used to obtain the known result that if \( S \) and \( W \) are continua and \( W \subseteq S \), then \( T(W) \) is a continuum \([1]\).

Proof of Theorem A. It is immediate from the definition that whenever \( A \subseteq B \), \( T(A) \subseteq T(B) \) and thus \( T(\bigcap \Gamma) \subseteq \bigcap \{ T(G) \mid G \in \Gamma \} \).

Suppose \( p \in T(\bigcap \Gamma) \). There exists \( W \), a subcontinuum of \( S \), such that \( p \in \text{Int}(W) \) and \( W \cap (\bigcap \Gamma) = \emptyset \). Since \( W \) is compact, there exists a finite collection \( G_1, \ldots, G_n \) of elements of \( \Gamma \) whose intersection is disjoint from \( W \). By hypothesis there exists \( G \), an element of \( \Gamma \), which is contained in \( G_1 \cap \cdots \cap G_n \). Since \( G \) is disjoint from \( W \), \( p \notin T(G) \). Hence \( p \notin \bigcap \{ T(G) \mid G \in \Gamma \} \) and thus

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Proof of Theorem B. The necessity of the condition is clear. Let \( \Lambda \) be a collection of closed subsets of \( S \) whose union is closed in \( S \). Since \( T(\cup \Lambda) \supseteq \bigcup \{ T(L) \mid L \in \Lambda \} \), it need only be shown that \( T(\cup \Lambda) \subseteq \bigcup \{ T(L) \mid L \in \Lambda \} \) in order to establish the sufficiency of the condition.

Suppose \( x \in \bigcup \{ T(L) \mid L \in \Lambda \} \). Then for each \( L \in \Lambda \) let \( F(L) \) be the collection of closed subsets of \( S \) such that \( L \subseteq \text{Int}(A) \). If \( L = \emptyset \), clearly \( T(L) = \bigcap \{ T(A) \mid A \in F(L) \} \). If \( L \neq \emptyset \), then \( F(L) \) is a filterbase of closed subsets of \( S \) and, since \( \bigcap F(L) = L \), \( T(L) = \bigcap \{ T(A) \mid A \in F(L) \} \) by Theorem A.

Hence, for each \( L \), \( x \in \bigcap \{ T(A) \mid A \in F(L) \} \) and thus there exists, for each \( L \), \( f(L) \in F(L) \), such that \( x \in \text{Int}(f(L)) \). \( \{ \text{Int}(f(L)) \mid L \in \Lambda \} \) is an open covering of \( \cup \Lambda \). Since \( \cup \Lambda \) is compact there exists a finite subcollection \( \Gamma \) of \( \{ \text{Int}(f(L)) \mid L \in \Lambda \} \) such that \( \cup \Lambda \subseteq \bigcup \Gamma \). Since, by hypothesis and induction \( T(\cup \Gamma) = \bigcup \{ T(G) \mid G \in \Gamma \} \), \( T(\cup \Lambda) \subseteq \bigcup \{ T(G) \mid G \in \Gamma \} \). Since for all \( G \in \Gamma \), \( x \in \text{Int}(G) \), it follows that \( x \in T(\cup \Lambda) \). Thus \( T(\cup \Lambda) \subseteq \bigcup \{ T(L) \mid L \in \Lambda \} \).

Proof of Theorem C. Two technical lemmas are established. Theorem C follows easily from these two lemmas and Theorem A.

Lemma 1. Let \( A \) be a subset of \( S \). \( p \in S - T(A) \) iff there is a subcontinuum \( W \) and an open subset \( Q \) of \( S \) such that \( p \in \text{Int}(W) \cap Q \), \( \text{Fr}(Q) \cap T(A) = \emptyset \) and \( W \cap A \cap Q = \emptyset \).

Proof. Let \( p \in S - T(A) \). There is a subcontinuum \( W \) of \( S \) such that \( p \in \text{Int}(W) \) and \( W \cap A = \emptyset \). Since \( S \) is regular there is an open subset \( Q \) of \( S \) such that \( p \in Q \) and \( \text{Cl}(Q) \subseteq \text{Int}(W) \). It is clear that \( \text{Fr}(Q) \cap T(A) = \emptyset \) and \( W \cap A \cap Q = \emptyset \).

Now suppose that there is a subcontinuum \( W \) and an open subset \( Q \) of \( S \) such that \( p \in \text{Int}(W) \cap Q \), \( \text{Fr}(Q) \cap T(A) = \emptyset \) and \( W \cap A \cap Q = \emptyset \). Since \( \text{Fr}(Q) \) is compact and disjoint from \( T(A) \), there exists a finite collection \( \{ W_i \} \) of subcontinua of \( S \), all disjoint from \( A \), such that \( \bigcup \{ \text{Int}(W_i) \} \supseteq \text{Fr}(Q) \). Since \( W \subseteq Q \) it is immediate that \( p \in S - T(A) \), assume \( W \cap S - Q \neq \emptyset \). The closure of each component of \( W \cap Q \) must intersect at least one of the \( W_i \)'s, since \( \text{Fr}(Q) \subseteq \bigcup \{ W_i \} \). Hence \( (W \cap Q) \cup (\bigcup \{ W_i \}) = H \) has only a finite number of components. Since \( p \in \text{Int}(W) \cap Q \), there is a component \( K \) of \( H \) such that \( p \in \text{Int}(K) \) and, of course, \( K \cap A \cap H \cap A = \emptyset \). Thus \( p \in S - T(A) \).

Lemma 2. Let \( A \) be a subset of \( S \). If \( T(A) = M \cup N \) separate then \( T(A \cap M) = M \cup T(\emptyset) \).
Proof. Suppose \( p \in T(A \cap M) - (M \cup T(\emptyset)) \). Since \( p \in T(\emptyset) \), there is a subcontinuum \( W \) of \( S \) such that \( p \in \text{Int}(W) \). Since \( S \) is normal, there is an open subset \( Q \) of \( S \) containing \( N \) whose closure is disjoint from \( M \). It is clear that \( p \in \text{Int}(W) \cap Q \), \( \text{Fr}(Q) \cap T(A \cap M) \subset \text{Fr}(Q) \cap T(A) = \emptyset \) and \( W \cap (A \cap M) \cap Q \subset Q \cap M = \emptyset \). Hence, by Lemma 1, \( p \in T(A \cap M) \), thus contradicting the supposition.

Now suppose that \( p \in (M \cup T(\emptyset)) - T(A \cap M) \). Since \( p \in T(A \cap M) \) and \( \emptyset \subset A \cap M, p \in T(\emptyset) \). Hence \( p \in M \). There is an open subset \( Q \) of \( S \) containing \( M \) whose closure is disjoint from \( N \). Since \( p \in T(A \cap M) \), there is a subcontinuum \( W \) of \( S \) such that \( p \in \text{Int}(W) \) and \( W \cap (A \cap M) = \emptyset \). It is clear that \( p \in \text{Int}(W) \cap Q \) and \( \text{Fr}(Q) \cap T(A) = \emptyset \). Since \( Q \cap N = \emptyset, W \cap A \cap Q = W \cap (A \cap M) = \emptyset \). Hence, by Lemma 1, \( p \in T(A) \) so \( p \in M \), thus contradicting the supposition.

Now in order to establish Theorem C, let \( A \) be a closed subset of \( S \) and \( K \) be a component of \( T(A) \). Let \( \{K_\alpha\} \) be the collection of all subsets of \( T(A) \) such that \( K \subset K_\alpha \) and \( K_\alpha \) is both open and closed in \( T(A) \). Note that the collection \( \{A \cap K_\alpha\} \) can only fail to be a filterbase if for some \( K_\alpha, A \cap K_\alpha = \emptyset \). In this case the conclusion of Theorem A is trivial. Lemma 2, of course, remains true even if \( A \cap M = \emptyset \) so, for each \( K_\alpha, T(A \cap K_\alpha) = K_\alpha \cup T(\emptyset) \). That this can occur is seen by letting \( S \) be the Cantor set, \( A \) be the void set and \( K_\alpha \) be \( S \).

The following sequence of equalities establish the theorem:

\[
T(A \cap K) = T(\bigcap \{A \cap K_\alpha\}) = \bigcap \{T(A \cap K_\alpha)\} = \bigcap \{K_\alpha \cup T(\emptyset)\} = \bigcap \{K_\alpha\} \cup T(\emptyset) = K \cup T(\emptyset).
\]

Theorem C is not true if the requirement that \( A \) be closed is dropped. Let \( S \) be the unit interval and let \( A \) be the sequence \( \{1/n\} \). Then \( T(A) = \{0\} \cup A \). Let \( K = \{0\} \). Then \( T(A \cap K) = T(\emptyset) \) which is void since \( S \) is a continuum. But \( K \cup T(\emptyset) \) is not void.

**Corollary 1.** Let \( S \) be a continuum and \( W \) be a subcontinuum of \( S \). \( T(W) \) is a subcontinuum of \( S \).

**Proof.** Suppose \( T(W) = A \cup B \) separate. By Theorem C, \( T(W \cap A) = A \) and \( T(W \cap B) = B \) since \( T(\emptyset) = \emptyset \) when \( S \) is a continuum. \( W \cap A \neq \emptyset \) since \( T(W \cap A) \neq \emptyset \) and, likewise \( W \cap B \neq \emptyset \). Hence \( W = (W \cap A) \cup (W \cap B) \) separate, contradicting the hypothesis and thus establishing the proposition.
Corollary 2. Let $S$ be a continuum and let $W_1$ and $W_2$ be subcontinua of $S$. If $T(W_1 \cup W_2) \neq T(W_1) \cup T(W_2)$ then $T(W_1 \cup W_2)$ is a continuum.

Proof. Suppose $T(W_1 \cup W_2) = A \cup B$ separate. By Lemma 2, $T((W_1 \cup W_2) \cap A) = A$ and $T((W_1 \cup W_2) \cap B) = B$. Suppose $W_1 \subset A$. If $W_2 \subset A$ then $A = T((W_1 \cup W_2) \cap A) = T(W_1 \cup W_2)$, thus contradicting the supposition. Hence $W_2 \subset B$. But then $T(W_1) = A$ and $T(W_2) = B$. Thus $T(W_1 \cup W_2) = T(W_1) \cup T(W_2)$. Corollaries 1 and 2 are special cases of Theorem 8 of [1].

Corollary 3. Let $S$ be a continuum and let $A$ and $B$ be closed subsets of $S$. If $K$ is a component of $T(A \cup B)$ which lies in neither $T(A)$ nor $T(B)$, then, $K \cap A \neq \emptyset \neq K \cap B$.

Proof. Since $S$ is a continuum, $T(\emptyset) = \emptyset$ and, by Theorem C, $T((A \cup B) \cap K) = K$. Since $K$ lies in neither $T(A)$ nor $T(B)$, $(A \cup B) \cap K$ meets both $A$ and $B$. Thus $K$ meets both $A$ and $B$.

Corollary 4. Let $S$ be a continuum and let $A$ and $B$ be closed subsets of $S$. If $T(A \cup B) \neq T(A) \cup T(B)$ then there exists a subcontinuum $K \subset T(A \cup B)$ such that $K \cap A \neq \emptyset \neq K \cap B$.

Proof. Let $K$ be the component of some point in $T(A \cup B) - (T(A) \cup T(B))$ and apply Corollary 3.

Bibliography


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