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CONTINUUM NEIGHBORHOODS AND FILTERBASES

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ABSTRACT. In this paper we prove that if Γ is a filterbase of closed subsets of a compact Hausdorff space then $T(\bigcap\Gamma) = \bigcap\{T(G) \mid G\in\Gamma\}$, where $T(A)$ denotes the set of those points for which every neighborhood which is a continuum intersects A nonvoidly.

Introduction. In this paper S denotes a compact Hausdorff space. If $p\in S$ and $W\subset S$, then W is a continuum neighborhood of p iff W is a subcontinuum of S and $p\in \operatorname{Int}(W)$. If $A\subset S$, $T(A)$ denotes the complement of the set of those points p of S for which there exists a continuum neighborhood which is disjoint from A [1]. S is said to be *T-additive* iff for every collection Λ of closed subsets of S whose union is closed, $T(\bigcup\Lambda) = \bigcup\{T(L) \mid L\in\Lambda\}$ [2]. The following three theorems are established.

THEOREM A. *Let Γ be a filterbase of closed subsets of S. Then* $T(\bigcap\Gamma) = \bigcap\{T(G) \mid G\in\Gamma\}$.

THEOREM B. *S is T-additive iff for each pair A, B of closed subsets of S, $T(A\cup B) = T(A)\cup T(B)$.*

THEOREM C. *Let A be a closed subset of S. If K is a component of $T(A)$ then $T(A\cap K) = K\cup T(\varnothing)$.*

Theorem A is used in establishing Theorems B and C. Theorem C is used to obtain the known result that if S and W are continua and $W\subset S$ then $T(W)$ is a continuum [1].

PROOF OF THEOREM A. It is immediate from the definition that whenever $A\subset B$, $T(A)\subset T(B)$ and thus $T(\bigcap\Gamma)\subset\bigcap\{T(G) \mid G\in\Gamma\}$.

Suppose $p\notin T(\bigcap\Gamma)$. There exists W, a subcontinuum of S, such that $p\in\operatorname{Int}(W)$ and $W\cap(\bigcap\Gamma) = \varnothing$. Since W is compact, there exists a finite collection $G_1, \cdots, G_n$ of elements of Γ whose intersection is disjoint from W. By hypothesis there exists G, an element of Γ, which is contained in $G_1\cap \cdots \cap G_n$. Since G is disjoint from W, $p\notin T(G)$. Hence $p\notin\bigcap\{T(G) \mid G\in\Gamma\}$ and thus

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$$T(\bigcap\Gamma)=\bigcap\{T(G)\mid G\in\Gamma).$$

Proof of Theorem B. The necessity of the condition is clear. Let Λ be a collection of closed subsets of S whose union is closed in S. Since $T(\bigcup\Lambda)\supset\bigcup\{T(L)\mid L\in\Lambda\}$, it need only be shown that $T(\bigcup\Lambda)\subset\bigcup\{T(L)\mid L\in\Lambda\}$ in order to establish the sufficiency of the condition.

Suppose $x\notin\bigcup\{T(L)\mid L\in\Lambda\}$. Then for each $L\in\Lambda$ let $F(L)$ be the collection of closed subsets A of S such that $L\subset\mathrm{Int}(A)$. If $L=\varnothing$, clearly $T(L)=\bigcap\{T(A)\mid A\in F(L)\}$. If $L\neq\varnothing$, then $F(L)$ is a filterbase of closed subsets of S and, since $\bigcap F(L)=L$, $T(L)=\bigcap\{T(A)\mid A\in F(L)\}$ by Theorem A.

Hence, for each L, $x\notin\bigcap\{T(A)\mid A\in F(L)\}$ and thus there exists, for each L, $f(L)\in F(L)$, such that $x\notin T(f(L))$. $\{\mathrm{Int}(f(L))\mid L\in\Lambda\}$ is an open covering of $\bigcup\Lambda$. Since $\bigcup\Lambda$ is compact there exists a finite subcollection Γ of $\{f(L)\mid L\in\Lambda\}$ such that $\bigcup\Lambda\subset\bigcup\Gamma$. Since, by hypothesis and induction $T(\bigcup\Gamma)=\bigcup\{T(G)\mid G\in\Gamma\}$, $T(\bigcup\Lambda)\subset\bigcup\{T(G)\mid G\in\Gamma\}$. Since for all $G\in\Gamma$, $x\notin T(G)$, it follows that $x\notin T(\bigcup\Lambda)$. Thus $T(\bigcup\Lambda)\subset\bigcup\{T(L)\mid L\in\Lambda\}$.

Proof of Theorem C. Two technical lemmas are established. Theorem C follows easily from these two lemmas and Theorem A.

Lemma 1. *Let A be a subset of S. $p\in S-T(A)$ iff there is a subcontinuum W and an open subset Q of S such that $p\in\mathrm{Int}(W)\cap Q$, $\mathrm{Fr}(Q)\cap T(A)=\varnothing$ and $W\cap A\cap Q=\varnothing$.*

Proof. Let $p\in S-T(A)$. There is a subcontinuum W of S such that $p\in\mathrm{Int}(W)$ and $W\cap A=\varnothing$. Since S is regular there is an open subset Q of S such that $p\in Q$ and $\mathrm{Cl}(Q)\subset\mathrm{Int}(W)$. It is clear that $\mathrm{Fr}(Q)\cap T(A)=\varnothing$ and $W\cap A\cap Q=\varnothing$.

Now suppose that there is a subcontinuum W and an open subset Q of S such that $p\in\mathrm{Int}(W)\cap Q$, $\mathrm{Fr}(Q)\cap T(A)=\varnothing$ and $W\cap A\cap Q=\varnothing$. Since $\mathrm{Fr}(Q)$ is compact and disjoint from $T(A)$, there exists a finite collection $\{W_i\}$ of subcontinua of S, all disjoint from A, such that $\bigcup\{\mathrm{Int}(W_i)\}\supset\mathrm{Fr}(Q)$. Since if $W\subset Q$ it is immediate that $p\in S-T(A)$, assume $W\cap S-Q\neq\varnothing$. The closure of each component of $W\cap Q$ must intersect at least one of the W_i's, since $\mathrm{Fr}(Q)\subset\bigcup\{W_i\}$. Hence $(W\cap Q)\cup(\bigcup\{W_i\})=H$ has only a finite number of components. Since $p\in\mathrm{Int}(W)\cap Q$, there is a component K of H such that $p\in\mathrm{Int}(K)$ and, of course, $K\cap A\subset H\cap A=\varnothing$. Thus $p\in S-T(A)$.

Lemma 2. *Let A be a subset of S. If $T(A)=M\cup N$ separate then $T(A\cap M)=M\cup T(\varnothing)$.*

PROOF. Suppose $p \in T(A \cap M) - (M \cup T(\varnothing))$. Since $p \notin T(\varnothing)$, there is a subcontinuum W of S such that $p \in \operatorname{Int}(W)$. Since S is normal, there is an open subset Q of S containing N whose closure is disjoint from M. It is clear that $p \in \operatorname{Int}(W) \cap Q$, $\operatorname{Fr}(Q) \cap T(A \cap M) \subset \operatorname{Fr}(Q) \cap T(A) = \varnothing$ and $W \cap (A \cap M) \cap Q \subset Q \cap M = \varnothing$. Hence, by Lemma 1, $p \notin T(A \cap M)$, thus contradicting the supposition.

Now suppose that $p \in (M \cup T(\varnothing)) - T(A \cap M)$. Since $p \notin T(A \cap M)$ and $\varnothing \subset A \cap M$, $p \notin T(\varnothing)$. Hence $p \in M$. There is an open subset Q of S containing M whose closure is disjoint from N. Since $p \notin T(A \cap M)$, there is a subcontinuum W of S such that $p \in \operatorname{Int}(W)$ and $W \cap (A \cap M) = \varnothing$. It is clear that $p \in \operatorname{Int}(W) \cap Q$ and $\operatorname{Fr}(Q) \cap T(A) = \varnothing$. Since $Q \cap N = \varnothing$, $W \cap A \cap Q = W \cap (A \cap M) = \varnothing$. Hence, by Lemma 1, $p \notin T(A)$ so $p \notin M$, thus contradicting the supposition.

Now in order to establish Theorem C, let A be a closed subset of S and K be a component of $T(A)$. Let $\{K_\alpha\}$ be the collection of all subsets of $T(A)$ such that $K \subset K_\alpha$ and K_α is both open and closed in $T(A)$. Note that the collection $\{A \cap K_\alpha\}$ can only fail to be a filterbase if for some K_α, $A \cap K_\alpha = \varnothing$. In this case the conclusion of Theorem A is trivial. Lemma 2, of course, remains true even if $A \cap M = \varnothing$ so, for each K_α, $T(A \cap K_\alpha) = K_\alpha \cup T(\varnothing)$. That this can occur is seen by letting S be the Cantor set, A be the void set and K_α be S.

The following sequence of equalities establish the theorem:

$$\begin{aligned} T(A \cap K) &= T(\cap\{A \cap K_\alpha\}) = \cap\{T(A \cap K_\alpha)\} \\ &= \cap\{K_\alpha \cup T(\varnothing)\} = \cap\{K_\alpha\} \cup T(\varnothing) \\ &= K \cup T(\varnothing). \end{aligned}$$

Theorem C is not true if the requirement that A be closed is dropped. Let S be the unit interval and let A be the sequence $\{1/n\}$. Then $T(A) = \{0\} \cup A$. Let $K = \{0\}$. Then $T(A \cap K) = T(\varnothing)$ which is void since S is a continuum. But $K \cup T(\varnothing)$ is not void.

COROLLARY 1. *Let S be a continuum and W be a subcontinuum of S. $T(W)$ is a subcontinuum of S.*

PROOF. Suppose $T(W) = A \cup B$ separate. By Theorem C, $T(W \cap A) = A$ and $T(W \cap B) = B$ since $T(\varnothing) = \varnothing$ when S is a continuum. $W \cap A \neq \varnothing$ since $T(W \cap A) \neq \varnothing$ and, likewise $W \cap B \neq \varnothing$. Hence $W = (W \cap A) \cup (W \cap B)$ separate, contradicting the hypothesis and thus establishing the proposition.

COROLLARY 2. *Let S be a continuum and let W_1 and W_2 be subcontinua of S. If $T(W_1 \cup W_2) \neq T(W_1) \cup T(W_2)$ then $T(W_1 \cup W_2)$ is a continuum.*

PROOF. Suppose $T(W_1 \cup W_2) = A \cup B$ separate. By Lemma 2, $T((W_1 \cup W_2) \cap A) = A$ and $T((W_1 \cup W_2) \cap B) = B$. Suppose $W_1 \subset A$. If $W_2 \subset A$ then $A = T((W_1 \cup W_2) \cap A) = T(W_1 \cup W_2)$, thus contradicting the supposition. Hence $W_2 \subset B$. But then $T(W_1) = A$ and $T(W_2) = B$. Thus $T(W_1 \cup W_2) = T(W_1) \cup T(W_2)$. Corollaries 1 and 2 are special cases of Theorem 8 of [1].

COROLLARY 3. *Let S be a continuum and let A and B be closed subsets of S. If K is a component of $T(A \cup B)$ which lies in neither $T(A)$ nor $T(B)$, then, $K \cap A \neq \emptyset \neq K \cap B$.*

PROOF. Since S is a continuum, $T(\emptyset) = \emptyset$ and, by Theorem C, $T((A \cup B) \cap K) = K$. Since K lies in neither $T(A)$ nor $T(B)$, $(A \cup B) \cap K$ meets both A and B. Thus K meets both A and B.

COROLLARY 4. *Let S be a continuum and let A and B be closed subsets of S. If $T(A \cup B) \neq T(A) \cup T(B)$ then there exists a subcontinuum $K \subset T(A \cup B)$ such that $K \cap A \neq \emptyset \neq K \cap B$.*

PROOF. Let K be the component of some point in $T(A \cup B) - (T(A) \cup T(B))$ and apply Corollary 3.

BIBLIOGRAPHY

1. H. S. Davis, D. P. Stadtlander and P. M. Swingle, *Properties of the set functions T^n*, Portugal. Math. **21** (1962), 113–133. MR **25** #5501.

2. H. S. Davis, *A note on connectedness in kleinen*, Proc. Amer. Math. Soc. **19** (1968), 1237–1241.

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