QUASI-JORDANIAN CONTINUA

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Abstract. A generalization of ordinary closed Jordan curves is obtained by considering nondegenerate compact continua which form the boundary of a simply connected open subset $D$ of Moore space such that all crosscuts of $D$ disconnect its closure. Separation in such continua by points of accessibility and the relation of such continua to their (possibly infinitely many) complementary domains are studied.

Introduction. Suppose that $M$ is a nondegenerate compact continuum which bounds a simply connected domain $D$, all of whose crosscuts disconnect $D$. We show that such continua have a connectivity structure much like that of closed Jordan curves. However, even in the Euclidean plane, such continua may possess infinitely many complementary domains.

Definitions and notation. $S$ will denote a space satisfying R. L. Moore's Axioms 0, 1, 2, 3, 4, and 5 [1]. If $X$ and $Y$ are point sets: $X \cup Y$ denotes their union; $X \cap Y$ denotes their common part; $\overline{X}$ is the closure of $X$ (in $S$); and $\text{Bd}(X)$ denotes the boundary of $X$. A complementary domain of a closed subset $K$ of $S$ is a component of $S - K$. A crosscut [endcut] of a connected open subset $D$ of $S$ is an arc both of whose ends [only one of whose ends] lie on the boundary of $D$, and whose remaining points are points of $D$. Any point of $\text{Bd}(D)$ that is an endpoint of an endcut of $D$ is said to be accessible from $D$. A domain $D$ is simply connected if and only if it is connected and contains a complementary domain of every simple closed curve lying in it. A compact nondegenerate continuum $M$ will be called quasi-Jordanian if and only if there exists a simply connected domain $D$ such that (1) $\text{Bd}(D) = M$ and (2) every crosscut of $D$ disconnects $\overline{D}$.

Conventions. For simplicity, the letter $M$ will consistently be employed to denote a quasi-Jordanian continuum lying in $S$ and $D$ will denote a complementary domain of $M$ satisfying (1) and (2).

Theorem 1. If $0$ is a point of $M$ and $J$ is a simple closed curve lying in $D \cup 0$, then $D$ contains a complementary domain of $J$. 

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Proof. The conclusion is automatic if $J$ does not pass through $0$. Suppose, however, that $0$ is a point of $J$ and that $D$ contains neither complementary domain of $J$. Then $J$ separates two points of $M$ from each other. Letting $I$ and $E$ denote the two complementary domains of $J$, it may be shown that there exist points $A \in I \cap \text{Bd}(D)$ and $B \in E \cap \text{Bd}(D)$ and a crosscut $AXB$ of $D$ such that $J \cap AXB = \emptyset$. Now any point belonging to $(I \cap D) - AX$ may be joined to $J - X$ by an arc lying in $D$ and missing $AXB$, since $D - AX$ is connected [1, p. 173, Theorem 20]. A similar process may be carried out for points of $(E \cap D) - XB$. It follows that $[D - AXB] \cup (J - X)$ is connected. But the latter set is dense in $\overline{D} - AXB$, contradicting the fact that $AXB$ disconnects $D$. Hence $D$ contains either $I$ or $E$.

Theorem 2. There exists only one complementary domain $E$ of $M$, distinct from $D$, whose boundary is the whole of $M$.

Proof. Suppose $M$ has a cutpoint $0$. Then there exists a simple closed curve $J$ such that (1) $M \cap J = \emptyset$ and (2) $J$ separates two points of $M$ from each other [1, p. 202, Theorem 53]. But then $J$ lies in $D \cap 0$, contradicting Theorem 1. Hence $M$ has no cutpoint. Now let $AB$ be a crosscut of $D$. Since $AB$ disconnects $D$, it may be seen that $\overline{D} - AB$ is the sum of two mutually separated connected sets $P$ and $Q$ such that $\overline{P} \cap \overline{Q} = AB$. With the aid of the latter equation and the observation that $AB$ does not separate $S$ [1, p. 175, Theorem 21], it is clear that $A \cup B$ separates two points $C$ and $F$ from each other in $M$. Furthermore, since $M$ has no cutpoint, $A \cup B$ is irreducible with respect to the property of separating $C$ from $F$ in $M$. Hence there exists a simple closed curve $J$ such that (1) $J$ separates $C$ from $F$ and (2) $J \cap M = A \cup B$ [1, p. 202, Theorem 53]. Denote by $AXB$ and $AYB$ the two arcs on $J$ with endpoints as indicated. An application of Theorem 1 shows that $D$ does not contain $J - (A \cup B)$ and since $J$ separates two points of $M$ from each other, it is clear that $S - \overline{D}$ does not contain $J - (A \cup B)$. Hence we may suppose that one of the two arcs $AXB$ and $AYB$, say $AXB$, lies, with the exception of its ends, wholly in $D$, and the other, say $AYB$, lies, with the exception of its ends, wholly in $S - \overline{D}$. Let $E$ denote the complementary domain of $M$ containing $Y$. Note that $A$ and $B$ are accessible points of $M$ lying on $\text{Bd}(E)$. Now let $A_1$ and $B_1$ denote any two points of $M$ which are accessible from $D$ and lie in distinct complementary domains of $J$. Then there exists an arc $A_1Y_1B_1$ such that $S - \overline{D}$ contains all of $A_1Y_1B_1$ except its ends. Clearly, $A_1Y_1B_1$ and $AYB$ intersect. Hence $A_1$ and $B_1$ lie on $\text{Bd}(E)$. It follows that each point of $M$ which is accessible from $D$ is a point of $\text{Bd}(E)$. Hence $\text{Bd}(E) = M$. Clearly, $E$ is the only
component of $S - \overline{D}$ with this property. For if $E'$ is any other, then the connected set $C \cup E' \cup F$ intersects both complementary domains of $J$. Hence $E'$ intersects $E$.

**Example.** On p. 119 of [2], there is an illustration of a connected planar region which spirals around a circular disk. The boundary $M$ of this region is quasi-Jordanian and has three (3) complementary domains. Other examples are easily obtainable of quasi-Jordanian continua with infinitely many complementary domains.

**Theorem 3.** No proper subcontinuum of $M$ disconnects $M$. In particular, $M$ has no cutpoint.

**Proof.** This is immediate since by Theorem 2, $M$ is the outer boundary of $D$ relative to $E$ [1, p. 178, Theorem 128].

**Theorem 4.** If $A$ and $B$ are points of $M$ which are accessible from $D$, then $M - (A \cup B)$ is the sum of two mutually separated connected sets $H$ and $K$ such that $H = H \cup A \cup B$ and $K = K \cup A \cup B$.

**Proof.** Consider a crosscut $AB$ of $D$. Then $\overline{D} - AB$ is the sum of two mutually separated connected sets $P$ and $Q$ such that $P \cap Q = AB$. Let $H = \text{Bd}(P) - AB$ and $K = \text{Bd}(Q) - AB$. Clearly, $H$ and $K$ are mutually separated subsets of $M$ and $M - (A \cup B) = H \cup K$. Theorem 3 implies that $H$ and $K$ are connected. Finally, since $M$ has no cutpoint, $H = H \cup A \cup B$ and $K = K \cup A \cup B$.

**Theorem 5.** The continua $H \cup A \cup B$ and $K \cup A \cup B$ are irreducible from $A$ to $B$.

**Proof.** This is an immediate consequence of Theorems 3 and 4.

**Theorem 6.** If $C$ is a point of the set $H$ which is accessible from $D$, then $H - C$ is the sum of two mutually separated connected sets $U$ and $V$ such that $U = U \cup A \cup C$ and $V = V \cup B \cup C$.

**Proof.** By Theorem 4, $M - (A \cup C)$ is the sum of two mutually separated connected sets $U$ and $W_1$ such that $U = U \cup A \cup C$ and $W_1 = W_1 \cup B \cup C$. Suppose for convenience that $W_1$ contains the connected set $K \cup B$. Then $U$ is a connected subset of $H - C$. Also, $M - (B \cup C)$ is the sum of two mutually separated connected sets $V$ and $W_2$ such that $V = V \cup B \cup C$ and $W_2 = W_2 \cup B \cup C$, and we suppose that $W_2$ contains the connected set $K \cup U$. Hence $U$ and $V$ are two mutually separated connected subsets of $H - C$ and $A \cup U \cup C \cup V \cup B$ is a continuum. By Theorem 5, $H \cup A \cup B = A \cup U \cup C \cup V \cup B$. Hence $H - C = U \cup V$.

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Theorem 7. If $A$, $B$, $C$, and $F$ are four points of $M$ which are accessible from $D$ and $A \cup B$ separates $C$ from $F$ in $M$, then $C \cup F$ separates $A$ from $B$ in $M$.

Proof. This is an immediate consequence of Theorems 4 and 6.

Theorem 8. If every point of $M$ is accessible from $D$, then $M$ is a simple closed curve.

Proof. By Theorem 4, each pair of points of $M$ disconnects $M$. Hence $M$ is a simple closed curve.

Theorem 9. If $M_1$ is the boundary of a complementary domain of $M$ distinct from both $D$ and $E$, then $M_1$ is a continuum of condensation of $M$.

Proof. Suppose, to the contrary, that $(M - M_1)$ does not contain $M_1$. Then there exist two points $A$ and $B$ of $M_1$ which are accessible from $D$. Let $H$ and $K$ denote the two components of $M - (A \cup B)$. By the proof of Theorem 2, either $H$ contains $M_1$ or $K$ contains $M_1$. Assume that $H = H \cup A \cup B$ contains $M_1$. As a consequence of our original assumption, there exist two points $C$ and $F$ belonging to $H \cap M_1$ which are accessible from $D$. By Theorems 4 and 6, $H - (C \cup F)$ is the sum of three mutually separated connected sets, $U$, $W$, and $V$, such that $U = U \cup A \cup C$, $W = W \cup C \cup F$, and $V = V \cup F \cup B$. But since $M_1$ does not intersect both components of $M - (C \cup F)$, it follows that $M_1$ is a subset of $U \cup V$. This is a contradiction, for the latter sets are mutually separated.

References


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