PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, No. 2, February 1971

AN EXTENSION OF THE NAGUMO UNIQUENESS THEOREM

J. M. BOWNDS AND F. T. METCALF

ABSTRACT. The classical Nagumo uniqueness theorem is a best possible result in the sense that if the Nagumo constant is replaced by a number greater than one then the result is false. This classical result uses the continuity of the right-hand side, $f(x, y)$, of the first order ordinary differential equation, and there is no explicit connection shown between the constant and the continuity of f; the observation that one makes is that the counterexample originally given by Perron uses a discontinuity at the origin together with a constant greater than one to obtain an initial-value problem with many solutions.

This paper shows explicitly that the original theorem remains true with Nagumo constant greater than one provided that f is sufficiently small at the origin, this sufficiency being determined by the actual value of the constant. Moreover, it is shown that the Nagumo inequality need not be imposed on the entire right-hand side of the equation; it suffices that only a certain factor satisfy a weakened form of the inequality.

1. **Introduction.** Let R denote the reals, and let $S=(x_0, x_1)\times R$. Throughout, it will be supposed that f is a real-valued function, $f\colon S\to R$. Any further restrictions on f will be explicitly listed as needed. Consider the initial-value problem

$$\text{(I)}\qquad \begin{aligned} &\text{(i)}\ \ y'(x) = f(x, y(x)), \qquad x\in(x_0, x_1),\\ &\text{(ii)}\ y(x_0) = y_0\in R. \end{aligned}$$

A function $y\colon [x_0, x_1)\to R$ is a solution to (I) if y' exists everywhere on (x_0, x_1) and satisfies (i), and $y\in C\{[x_0, x_1)\}$ and satisfies (ii).

The only question to be considered here is that of uniqueness for (I), and, in particular, a theorem will be proved below which extends the Nagumo uniqueness theorem [1] while illuminating a certain connection between the original hypotheses.

2. **The Nagumo theorem.** Nagumo's original theorem states that if $f\in C\{[x_0, x_1)\times R\}$ and satisfies

$$(1)\qquad (x-x_0)\,|f(x, y_2)-f(x, y_1)| \leqq |y_2-y_1|,$$

Received by the editors November 28, 1969.

AMS 1969 *subject classifications*. Primary 3404.

Key words and phrases. Uniqueness theorem for ordinary differential equations.

for $x \in (x_0, x_1)$ and $y_1, y_2 \in R$, then (I) has at most one solution. An example of Perron [2] shows that (1) cannot be replaced by

$$(2) \qquad (x - x_0) \mid f(x, y_2) - f(x, y_1) \mid \leqq (1 + \epsilon) \mid y_2 - y_1 \mid,$$

if $\epsilon > 0$.

Diaz and Walter [3], while retaining (1), removed the hypothesis of continuity on f, and replaced it with the assumption that f be continuous at (x_0, y_0). (See also another paper of Diaz [4].) Furthermore, that paper provides an example to show the necessity of the existence of

$$(3) \qquad \lim_{(x,y)\to(x_0^+, {}_0y)} f(x, y).$$

As a more general theorem below will show, it is possible to replace (1) by the one-sided inequality

$$(4) \qquad (x - x_0)(f(x, y_2) - f(x, y_1)) \leqq \alpha(y_2 - y_1)$$

where $\alpha \geqq 0$ is a constant, provided $x \in (x_0, x_1)$, $y_2 > y_1$, and the following limit exists

$$(5) \qquad \lim_{(x,y)\to(x_0^+, y_0)} (x - x_0)^{1-\alpha} f(x, y).$$

This one-sided theorem is, also, a clear generalization of the one-dimensional case of the classical one-sided Nagumo theorem, see Hartman [5, p. 35].

3. **A more general theorem of the Nagumo type.** The following uniqueness theorem contains, as a special case, the above mentioned theorem which utilizes (4) and (5). This theorem assumes a certain factorization of f, which has been used to obtain another uniqueness result [6].

THEOREM. *Suppose there exists a circular neighborhood N of (x_0, y_0) of radius $\delta > 0$, and there exist $f_1, f_2, f_3 : N \cap \bar{S} \to R \ni f \equiv f_1 f_2 - f_3$ on $N \cap \bar{S}$, where $f_1, f_3 \in C\{N \cap \bar{S}\}$ and $\partial f_1/\partial x$ exists, is nonnegative, and dominated by an integrable function of y on $N \cap \bar{S}$. Further, suppose that on $N \cap \bar{S}$:*

(i) *$f_1(x, y)$ is strictly positive,*

(ii) *$f_3(x, y)/f_1(x, y)$ is nondecreasing with respect to y,*

(iii) *$\exists \alpha \geqq 0 \ni f_2$ satisfies (4) and*

$$\lim_{(x,y)\to(x_0^+, y_0)} \left[\frac{f_2(x, y)}{(x - x_0)^{M\alpha - 1}}\right] \textit{exists},$$

where $M = \max | f_1(x, y) |$ on $\bar{N} \cap \bar{S}$.

Then there is at most one solution to (I).

REMARK. By considering explicit examples of nonuniqueness, it is possible to see that assumptions (i), (ii), and (iii) cannot be omitted. Also, it is clear that (5) is a special case of the limit appearing in (iii).

PROOF OF THE THEOREM. If u_1, u_2 are both solutions to (I), then it is a trivial matter to show that

$$m(x) = \max_{x \in [x_0, x_1]} \{u_1(x), u_2(x)\}$$

is also a solution.

Now, $f_1(x, y)$ is bounded away from zero on $\overline{N} \cap \overline{S}$; hence, by taking a sufficiently small interval $[x_0, x_0+\delta'] \subseteq [x_0, x_0+\delta]$, one may define the continuous functions H_i: $[x_0, x_0+\delta'] \to R$ by

$$\begin{aligned} H_i(x) = &\int_{u_i(x)}^{m(x)} [f_1(x, y)]^{-1}\, dy \\ &+ \int_{x_0}^{x} \left(\frac{f_3(y, m(y))}{f_1(y, m(y))} - \frac{f_3(y, u_i(y))}{f_1(y, u_i(y))} \right) dy. \end{aligned} \tag{7}$$

Clearly, $H_i(x_0) = 0$, and, using (i) and (ii),

$$H_i(x) \geqq 0 \quad \text{on } (x_0, x_0 + \delta']. \tag{8}$$

Further, there are sufficient conditions on f_1 and f_3 to imply the differentiability of H_i, and, in fact

$$H_i'(x) \leqq \frac{m'(x) + f_3(x, m(x))}{f_1(x, m(x))} - \frac{u_i'(x) + f_3(x, u_i(x))}{f_1(x, u_i(x))}; \tag{9}$$

or, using the factorization for f, together with (4),

$$H_i'(x) \leqq \frac{\alpha[m(x) - u_i(x)]}{x - x_0}. \tag{10}$$

Finally, using the bound M on f_1, together with (7),

$$H_i'(x) - \frac{M\alpha H_i(x)}{x - x_0} \leqq 0 \quad \text{on } (x_0, x_0 + \delta'), \qquad i = 1, 2. \tag{11}$$

Now, for $0 < \epsilon < \delta'$, (11) implies that, on $(x_0+\epsilon, x_0+\delta')$:

$$\frac{d}{dx} \left\{ -H_i(x) \exp\left[-M\alpha \int_{x_0+\epsilon}^{x} (s - x_0)^{-1}\, ds \right] \right\} \geqq 0; \tag{12}$$

hence, the function to be differentiated in this expression is nondecreasing, which implies

$$\frac{H_i(x)}{(x-x_0)^{M\alpha}} \leqq \frac{H_i(x_0+\epsilon)}{\epsilon^{M\alpha}} \quad \text{on } [x_0+\epsilon,\, x_0+\delta']. \tag{13}$$

If $\alpha = 0$, (13) implies that $H_i(x) \leqq 0$ on $(x_0, x_0+\delta']$, which, together with (8), implies that $H_i(x) \equiv 0$, that is, $m(x) \equiv u_1(x) \equiv u_2(x)$. If $\alpha > 0$, then $\exists \eta \ni 0 < \eta < \epsilon$ and

$$0 \leqq \frac{H_i(x_0+\epsilon)}{\epsilon^{M\alpha}} = \frac{H_i'(x_0+\eta)}{M\alpha\eta^{M\alpha-1}}. \tag{14}$$

But, using the factorization of f, together with (10), one has

$$H_i'(x_0+\eta) \leqq f_2(x_0+\eta,\, m(x_0+\eta)) - f_2(x_0+\eta,\, u_i(x_0+\eta)), \tag{15}$$

and, since (iii) implies the existence of

$$\lim_{\eta\to 0^+} \frac{f_2(x_0+\eta,\, u_i(x_0+\eta))}{\eta^{M\alpha-1}}, \tag{16}$$

it follows that

$$0 \leqq \frac{H_i(x)}{(x-x_0)^{M\alpha}} \leqq \lim_{\epsilon\to 0^+} \frac{H_i(x_0+\epsilon)}{\epsilon^{M\alpha}} = 0 \quad \text{on } [x_0,\, x_0+\delta');$$

that is, $H_i(x) \equiv 0$, so that $u_1(x) \equiv u_2(x)$ as before.

References

1. M. Nagumo, *Eine hinreichende Bedingung für die Unität der Lösung von Differentialgleichungen erster Ordnung*, Japan J. Math. **3** (1926), 107–112.

2. O. Perron, *Eine hinreichende Bedingung für die Unität der Lösung von Differentialgleichungen erster Ordnung*, Math. Z. **28** (1928), 216–219.

3. J. B. Diaz and W. L. Walter, *On uniqueness theorems for ordinary differential equations and for partial differential equations of hyperbolic type*, Trans. Amer. Math. Soc. **96** (1960), 90–100. MR **22** #11205.

4. J. B. Diaz, *On existence, uniqueness, and numerical evaluation of solutions of ordinary and hyperbolic differential equations*, Ann. Mat. Pura Appl. (4) **52** (1960), 163–181. MR **22** #11206.

5. P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964. MR **30** #1270.

6. J. Bownds, *A uniqueness theorem for $y' = f(x, y)$ using a certain factorization of f*, J. Differential Equations **7** (1970), 227–231.

University of Arizona, Tucson, Arizona 85721

University of California, Riverside, California 92502