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THE ALMOST FIXED POINT PROPERTY FOR HEREDITARILY UNICOHERENT CONTINUA

WILLIAM J. GRAY AND LOY O. VAUGHAN

ABSTRACT. It is shown that a hereditarily unicoherent Hausdorff continuum X has the almost fixed point property with respect to continuum valued mappings and finite coverings by subcontinua of X.

In [3], de Groot, de Vries, and Van der Walt defined the *almost fixed point property* as follows: let X be a space, let F be a collection of functions from X into X, and Δ be a collection of coverings of X. Then X has the almost fixed point property with respect to F and Δ if for each function f in F and covering α in Δ, there is an element A of α such that A meets fA. Although it was apparently intended that the members of F be single valued mappings, in this paper we will stretch the definition a bit and allow the members of F to be multivalued functions.

A *continuum* is a compact connected space. A continuum X is *unicoherent* provided that if X is the union of two subcontinua A and B, then $A \cap B$ is a continuum. X is *hereditarily unicoherent* if every subcontinuum of X is unicoherent. A *continuum valued function* on a continuum X is a multivalued function f which assigns to each x in X a subset $fx \subset X$ such that fA is a continuum for each subcontinuum A of X. Here $fA = \bigcup \{fx : x \in A\}$. A multivalued function on X is *upper semicontinuous* if given a neighborhood V of fx, there is a neighborhood U of x such that $fU \subset V$. An upper semicontinuous multivalued function on X which sends points to continua is a continuum valued function in the sense of the above definition.

Hopf [4] proved that a locally connected unicoherent metric continuum has the almost fixed point property with respect to continuous mappings and finite closed coverings of order 2. In [3] it is shown that

(1) E^2 has the almost fixed point property with respect to continuous mappings and finite coverings by convex open sets,

(2) E^2 has the almost fixed point property with respect to orientation preserving topological isometries and finite coverings by arcwise connected sets, and

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(3) a unicoherent space has the almost fixed point property with respect to continuous mappings and coverings consisting of three connected open sets.

In this paper it will be shown that a hereditarily unicoherent Hausdorff continuum has the almost fixed point property with respect to continuum valued mappings and finite subcovers by subcontinua of X.

In [5], Wallace generalized the Scherrer fixed point theorem by showing that an upper semicontinuous multivalued mapping of a tree has a fixed point (a tree is a locally connected hereditarily unicoherent Hausdorff continuum); the Theorem of this paper implies this result. As a matter of related interest, we remark that it is known that an arcwise connected hereditarily unicoherent Hausdorff continuum has the fixed point property. (See Young [6]. This result was first proved for metric spaces by Borsuk [1].) It is not known if hereditarily unicoherent, hereditarily decomposable continua have the fixed point property.

Let α be a collection of subsets of a space X, and $A, B \in \alpha$. A *chain* in α from A to B is a sequence $A_1, A_2, \cdots, A_n$ in α such that $A_1 = A$, $A_n = B$, and each two consecutive sets of the sequences intersect. The chain is *simple* provided that terms that are not consecutive do not intersect. If α is a finite closed covering of a connected space and A, $B \in \alpha$, then there is a chain in α from A to B, and every such chain contains a simple chain from A to B. A subset C of X *links* two subsets A, B of X if C meets both A and B.

Note that if X is a hereditarily unicoherent Hausdorff continuum, then the intersection of an arbitrary collection of subcontinua of X is again a continuum. In what follows, this result will be used frequently and without explicit mention.

LEMMA 1. *Let X be a hereditarily unicoherent Hausdorff continuum and A, B, C, D be subcontinua of X. If A and B are disjoint and are linked by both C and D, then C meets D.*

PROOF. Suppose that $C \cap D$ is empty. Then the intersection of the continuum $A \cup B \cup C$ with D is $A \cap D \cup B \cap D$, both sets in the union being nonempty. But A and B are disjoint, so that $A \cup B \cup C \cup D$ is not unicoherent.

LEMMA 2. *A collection α of subcontinua of a hereditarily unicoherent Hausdorff continuum has the finite intersection property if and only if any two elements of α have a nonempty intersection.*

PROOF. See Lemma 1 of [2].

THEOREM. *A nonempty hereditarily unicoherent Hausdorff continuum X has the almost fixed point property with respect to continuum valued mappings and finite coverings by subcontinua of X.*

PROOF. Let α be a finite covering of the space X and f be a continuum valued mapping on X. We must show that there is an $A\in\alpha$ which meets fA. If $\mathrm{Card}(\alpha)=1$, we are finished. We proceed by induction on $\mathrm{Card}(\alpha)$, and assume that the Theorem is true for all finite coverings β of X by subcontinua of X for which $\mathrm{Card}(\beta)<\mathrm{Card}(\alpha)$. We assume that A and fA are disjoint for every A in α.

We prove that if A and B are two elements of α which have a nonempty intersection, then either

(1) $A\cap fB\neq\varnothing$ and $B\cap fA=\varnothing$, or

(2) $A\cap fB=\varnothing$ and $B\cap fA\neq\varnothing$.

Indeed the covering β of X having for elements the continuum $A\cup B$ and the continua of α distinct from A or B has one less element than α. By the inductive hypothesis and the assumption that C and fC are disjoint for every C in α, we have

$$(A\cup B)\cap f(A\cup B)\neq\varnothing.$$

This means that

$$(A\cap fB)\cup(B\cap fA)\neq\varnothing.$$

Consequently, the nonempty continuum $(A\cap fB)\cup(B\cap fA)$ is the union of two disjoint closed sets $A\cap fB$ and $B\cap fA$, and so one of (1) or (2) must hold.

Next we show that if $A_1, A_2, \cdots, A_n$ is a simple chain in α such that $A_n\cap fA_1\neq\varnothing$, then fA_1 meets A_2. For by (1)–(2), either fA_1 meets A_2 or fA_2 meets A_1. Assume that A_1 meets fA_2. Then $fA_1\cup A_n\cup\cdots\cup A_3$ and A_1 link the disjoint subcontinua A_2 and fA_2. Since

$$A_1\cap(fA_1\cup A_n\cup\cdots\cup A_3)=\varnothing,$$

this violates Lemma 1. Therefore A_1 and fA_2 are disjoint, and so A_2 meets fA_1.

We will now show that for each positive integer n, there is a sequence $A_1, \cdots, A_{n-1}, B_n$ of distinct elements of α and a sequence $\alpha_1, \cdots, \alpha_{n-1}$ of subcollections of α such that

(3) $A_i\in\alpha_i$, $i=1, \cdots, n-1$.

(4) $\alpha_k=\{A\in\alpha: A \text{ links } A_{k-1} \text{ and } fA_{k-1}\}$, $k=2, \cdots, n-1$.

(5) $\bigcup\{B: B\in\alpha_i, i<n\}\cap fB_n=\varnothing$.

(6) $A_1, \cdots, A_{n-1}, B_n$ is a simple chain in α.

Choose an arbitrary nonempty element A_1 of α and let $\alpha_1=\{A_1\}$.

Assume sequences which satisfy (3)–(6) have been defined for some integer $n \geqq 1$. Since B_n meets A_{n-1} and fB_n and A_{n-1} are disjoint, we deduce that B_n meets fA_{n-1}. In accordance with (4) we define

$$\alpha_n = \{A \in \alpha: A \text{ links } A_{n-1} \text{ and } fA_{n-1}\}.$$

If $A, B \in \alpha_n$, the disjoint subcontinua A_{n-1} and fA_{n-1} are linked by A and B so that by Lemma 1, A meets B. Then Lemma 2 implies that $\alpha_n \cup \{A_{n-1}\}$ has the finite intersection property. Consequently

$$H = (\bigcap \{A \in \alpha_n\}) \cap A_{n-1} \neq \emptyset.$$

Choose an element C in α which meets fH. Since $B_n \in \alpha_n$ and $fH \subset fB_n$, we deduce from (5) that $C \notin \alpha_i$ for $i \leqq n$ (note that A and fH are disjoint for every A in α_n since $A \cap fA$ is empty for every A in α_n). Let $C_1, \cdots, C_m = C$ be a simple chain in α of minimal length which joins C to an element C_1 in α_n, i.e. any simple chain in α from C to an element in α_n has at least m elements.

We show that $A_1, \cdots, A_{n-1}, C_1$ is a simple chain in α. If B_n meets C_2, we may take $B_n = C_1$ and the desired result follows from (6). If B_n and C_2 are disjoint, then $B_n, C_1, C_2, \cdots, C_m$ is a simple chain in α. Because fB_n meets C_m, we conclude that fB_n also meets C_1. On the other hand, the disjoint subcontinua A_{n-1} and fA_{n-1} are linked by C_1 and fA_{n-2}, so that Lemma 1 implies that C_1 meets fA_{n-2}. It follows that if C_1 meets A_{n-2}, then C_1 is an element of α_{n-1}. But then because of (5), C_1 could not meet fB_n, and this is a contradiction. Thus C_1 does not meet A_{n-2}. Suppose now that C_1 meets A_i for some $i < n-2$. Let r be the largest integer such that $A_r \cap C_1$ is not empty; r is less than $n-2$. Then the disjoint subcontinua C_1 and $A_{r+1} \cup \cdots \cup A_{n-2}$ link the disjoint subcontinua A_r and A_{n-1}, which contradicts Lemma 1. This shows that $A_1, \cdots, A_{n-1}, C_1$ is a simple chain.

We next show that $A_1, \cdots, A_{n-1}, C_1, C_2$ is a simple chain. To this end assume that C_2 meets A_{n-1}. Let k be the largest integer for which $A_{n-1} \cap C_k$ is not empty; then $k \geqq 2$, and $A_{n-1}, C_k, C_{k+1}, \cdots, C_m$ is a simple chain from A_{n-1} to C_m. Since fA_{n-1} meets C_m, we conclude that fA_{n-1} also meets C_k, from which it follows that C_k is in α_n. But then $C_k, C_{k+1}, \cdots, C_m$ is a simple chain from $C_m = C$ to an element C_k in α_n of length less than m, and we have a contradiction. Thus C_2 and A_{n-1} are disjoint, and as above, we can prove that $A_1, \cdots, A_{n-1}, C_1, C_2$ is a simple chain in α.

Define $A_n = C_1$. Note that since $A_n, C_2, \cdots, C_m$ is a simple chain and C_m meets fA_n, we have

$$fA_n \cap C_2 \neq \emptyset. \tag{7}$$

Let $B \in \alpha_n$. We prove that

$$fC_2 \cap B = \varnothing. \tag{8}$$

If B does not meet C_2, then B, A_n, C_2 is a simple chain from B to C_2. Hence if we assume that fC_2 meets B, we would conclude that fC_2 meets A_n, and (7) could not hold. This means that if B and C_2 are disjoint, then fC_2 does not meet B. Now assume that B meets C_2. If in addition B meets C_j for some $j>2$, then $B, C_j, \cdots, C_m$ would contain a simple chain from $C_m = C$ to an element B in α_n of length less than m, and this is not possible. Thus $B, C_2, \cdots, C_m$ is a simple chain. Since

$$fB \cap C_m \supset fH \cap C_m \neq \varnothing,$$

fB meets C_2. Then (1) and (2) imply that fC_2 does not meet B. This proves (8).

Let $B \in \alpha_i$ for $i < n$. We prove that

$$fC_2 \cap B = \varnothing. \tag{9}$$

If B does not meet C_2, let r be the largest integer for which $B \cap A_r$ is not empty; then $r \leqq n$ and $B, A_r, \cdots, A_n, C_2$ is a simple chain. If we assume that B meets fC_2, we conclude that fC_2 meets A_n, and this violates (8). This means that (9) holds when B and C_2 are disjoint. If B meets C_2, then from $B \in \alpha_i$ we also have that B meets A_i, and it is not difficult to see that B meets A_r for $i \leqq r \leqq n$. Suppose that $fC_2 \cap B$ is not empty. Then the disjoint subcontinua C_2 and fC_2 are linked by B and fA_n, and Lemma 1 implies that B meets fA_n. Consequently the disjoint subcontinua A_n and fA_n are linked by B and fA_{n-1} which implies that $B \cap fA_{n-1}$ is not empty. But then B links A_{n-1} and fA_{n-1} so that B is in α_n, and from what we have just shown, this implies that fC_2 and B are disjoint. Thus (9) follows.

In summary, we have shown that fC_2 and B are disjoint for every $B \in \alpha_i$ for $i \leqq n$. We now define $B_{n+1} = C_2$ and observe that $A_1, \cdots, A_n, B_{n+1}, \alpha_1, \cdots, \alpha_n$ satisfy (3)–(6) so that the proof of the inductive step is completed.

Finally, we observe that the above inductive construction is impossible since α is finite, and so there exists an A in α which meets fA. This completes the proof of the Theorem.

We remark that all the hypotheses of the Theorem were used except for compactness, which is inherent in the term "Continuum." We call a space X hereditarily unicoherent if any two closed connected subsets of X have connected intersection. Then an exact analogue of the above Theorem can be proved for hereditarily unicoherent spaces provided one talks about closed connected subsets rather than subcontinua. However, in reality this would amount merely to placing a new interpretation on the word continuum, and so no significant generalization of our Theorem would be obtained.

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References

1. K. Borsuk, *A theorem on fixed points*, Bull. Acad. Polon. Sci. Cl. III, **2** (1954), 17–20. MR **16**, 275.

2. William J. Gray, *A fixed point theorem for commuting monotone functions*, Canad. J. Math. **21** (1969), 502–504. MR **39** #942.

3. J. de Groot, H. de Vries and T. Van der Walt, *Almost fixed point theorems for the Euclidean plane*, Nederl. Akad. Wetensch. Proc. Ser. A **66** = Indag. Math. **25** (1963), 606–612. MR **27** #6256.

4. H. Hopf, *Freie überdeckungen and freie abildungen*, Fund. Math. **28** (1937), 33–57.

5. A. D. Wallace, *A fixed point theorem for trees*, Bull. Amer. Math. Soc. **47** (1941), 757–760. MR **3**, 57.

6. G. S. Young, *Fixed-point theorems for arcwise connected continua*, Proc. Amer. Math. Soc. **11** (1960), 880–884. MR **22** #8486.

University of Alabama, University, Alabama 35486