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SEMIADJOINT FUNCTORS AND QUINTUPLES

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ABSTRACT. We obtain a generalization of adjoint functors and triples by axiomatizing the behavior of Albanese varieties, obtain a few basic properties, and show that abelian varieties form a generalized type of triplable category over complete abstract varieties.

1. Basic properties. Let $U:\mathcal{B}\to\mathcal{A}$ be a functor. We will say that U has a *left semiadjoint* V if there are an object function $V:\mathrm{Obj}\,\mathcal{A}\to\mathrm{Obj}\,\mathcal{B}$ and a functor $E:\mathcal{B}\to\mathrm{Ab}$, where Ab is the category of abelian groups, such that $EB\subseteq\mathrm{Aut}(UB)$, and the following conditions are satisfied. For every object A of $\mathcal{A}$, there is a fixed map $\alpha_A:A\to UVA=FA$ such that $f:A\to UB$ implies the existence of unique maps $g:VA\to B$ and $c\in EB$ such that $f=cU(g)\alpha_A$. Furthermore, if $h:B_1\to B_2$ and $c\in EB_1$ then $E(h)(c)\circ U(h)=U(h)\circ c$. We will often write h_* or even $U(h)_*$ for $E(h)$. EB will be called the group of *translations* of B.

Note first that V becomes a functor in the usual way. Let $f:A\to B$, $g:B\to C$ be maps of $\mathcal{A}$; then $V(f)$ is the unique map in $\mathcal{B}$ such that $\alpha_B f=cUV(f)\alpha_A$ for some $c\in EVB$. We will denote this c by $c(f)$. Clearly $V(1_A)=1_{VA}$. Also $\alpha_C g=c(g)UV(g)\alpha_B$ and we have

$$\begin{aligned}\alpha_C gf &= c(g)UV(g)\alpha_B f\\ &= c(g)UV(g)c(f)UV(f)\alpha_A\\ &= c(g)(V(g))_*(c(f))U(V(g)V(f))\alpha_A\end{aligned}$$

whence $V(gf)=V(g)V(f)$ and V is a functor. Then $\alpha:1_{\mathcal{A}}\to UV=F$ is a *seminatural* transformation, that is, for each $f:A\to B$ there is a unique $c(f)\in EVB$ such that $\alpha_B f=c(f)F(f)\alpha_A$. We have also the result that $c(gf)=c(g)V(g)_*c(f)$.

We obtain a back adjunction $\beta:VU\to 1_{\mathcal{B}}$ in the usual way; if $B\in\mathrm{Obj}\,\mathcal{B}$ then $\beta_B:VUB\to B$ and $d_B\in EB$ are the unique maps such that $d_BU(\beta_B)\alpha_{UB}=1_{UB}$. Thus $U(\beta_B)\alpha_{UB}=d_B^{-1}\in EB$. Now suppose that $f:A\to UB$; the unique maps $g:VA\to B$ and $c\in EB$ such that $f=cU(g)\alpha_A$ are given by the formulas $g=\beta_B V(f)$ and $c=d_B(\beta_B)_*c(f)$, since

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$$\begin{aligned} d_B(\beta_B)_*c(f)U(\beta_B)F(f)\alpha_A &= d_BU(\beta_B)c(f)F(f)\alpha_A \\ &= d_BU(\beta_B)\alpha_{UB}f \\ &= f. \end{aligned}$$

In contrast to α, β is a true natural transformation: if $g:B\to C$ in $\mathfrak{B}$, $g\beta_B=\beta_C VU(g)$. To see this, we observe that

$$U(g\beta_B)\alpha_{UB} = U(g)\overset{-1}{d_B} = g_*(\overset{-1}{d_B})U(g)$$

and

$$\begin{aligned} U(\beta_C VU(g))\alpha_{UB} &= U(\beta_C)FU(g)\alpha_{UB} \\ &= U(\beta_C)c(Ug)^{-1}\alpha_{UC}U(g) \\ &= (\beta_C)_*c(Ug)^{-1}U(\beta_C)\alpha_{UC}U(g) \\ &= (\beta_C)_*c(Ug)^{-1}\overset{-1}{d_C}U(g) \end{aligned}$$

from which we get

$$U(g) = g_*(d_B)U(g\beta_B)\alpha_{UB}$$

and

$$U(g) = d_C(\beta_C)_*c(Ug)U(\beta_C VU(g))\alpha_{UB}$$

whence $\beta_C VU(g)=g\beta_B$; also, $g_*(d_B)=d_C\circ(\beta_C)_*c(Ug)$. Furthermore, β acts almost like an ordinary back adjunction. If $g\colon VA\to B$, there is at least one map $j:A\to UB$ such that $\beta_B V(j)=g$, namely $j=U(g)\alpha_A$. To see this, note that $\beta_B V(j)=\beta_B VU(g)V(\alpha_A)=g\beta_{VA}V(\alpha_A)$, so we need only show that $\beta_{VA}V(\alpha_A)=1_{VA}$. But

$$\begin{aligned} U(\beta_{VA}V(\alpha_A))\alpha_A &= U(\beta_{VA})F(\alpha_A)\alpha_A \\ &= U(\beta_{VA})c(\alpha_A)^{-1}\alpha_{FA}\alpha_A \\ &= (\beta_{VA})_*c(\alpha_A)^{-1}U(\beta_{VA})\alpha_{FA}\alpha_A \\ &= (\beta_{VA})_*c(\alpha_A)^{-1}\circ\overset{-1}{d_{VA}}\alpha_A \end{aligned}$$

so $\alpha_A=(\beta_{VA})_*(c(\alpha_A))d_{VA}U(\beta_{VA}V(\alpha_A))\alpha_A$ and this implies that $\beta_{VA}V(\alpha_A)=1_{VA}$ and also that $d_{VA}^{-1}=(\beta_{VA})_*c(\alpha_A)$.

Further miscellaneous consequences of the definitions which we shall need are: first, if $c\in EVUB$ then

$$\begin{aligned} (\beta_B)_*(c) &= (\beta_B)_*(c)1_{UB} = (\beta_B)_*(c)U(\beta_B)\alpha_{UB}d_B \\ &= \beta_B c\alpha_{UB}d_B. \end{aligned}$$

Next, let $c\in EB$; then

$$\begin{aligned}(\beta_B)_*c(c)U(\beta_B V(c))\alpha_{UB} &= U(\beta_B)c(c)F(c)\alpha_{UB}\\ &= U(\beta_B)\alpha_{UB}c\\ &= \overset{-1}{d_B}c = c\overset{-1}{d_B}\\ &= cU(\beta_B)\alpha_{UB}\end{aligned}$$

which implies that $\beta_B = \beta_B V(c)$ and also that $c = (\beta_B)_*c(c)$, i.e., $U(\beta_B)c(c) = cU(\beta_B)$. Finally, suppose that $d \in EB$, $f: A \to UB$, and $df = f$. Then there exist unique $c \in EB$ and $g: VA \to B$ such that $cU(g)\alpha_A = f$; then $dcU(g)\alpha_A = cU(g)\alpha_A$ implies $dc = c$, or $d = 1_{UB}$. Thus a translation other than 1 has no "fixed points."

Now we are ready to define the analog for semiadjoints of the concept of triple for ordinary adjoints.

DEFINITION. A *quintuple* $\mathcal{Q} = (F, \alpha, \mu, G, c)$ on a category $\mathcal{A}$ consists of a functor $F: \mathcal{A} \to \mathcal{A}$, a functor $G: \mathcal{A} \to \mathrm{Ab}$ such that $GA \subseteq \mathrm{Aut}(FA)$, a natural transformation $\mu: F^2 \to F$, and a seminatural transformation $\alpha: 1_{\mathcal{A}} \to F$ where, if $f: A \to B$, $\alpha_B f = c(f)F(f)\alpha_A$ for a unique $c(f)$ in GB; also, $\mu \circ F\alpha = 1$, $\mu \circ F\mu = \mu \circ \mu F$, and $\mu_A \alpha_{FA} \in GA$. We write d_{VA}^{-1} for $\mu_A \alpha_{FA}$. In addition, we require that: if $d \in GFA$, then $\mu_A d = \mu_A d \alpha_{FA} d_{VA} \mu_A$; if $c \in GA$ then $c\mu_A = \mu_A c(c)$ and $\mu_A F(c) = \mu_A$; and $f: B \to FA$, $cf = f$ imply $c = 1_{FA}$. An *algebra* over $\mathcal{Q}$ is a pair (A, ξ) where $\xi: FA \to A$, $\xi F(\xi) = \xi\mu_A$, and $\xi\alpha_A \in \mathrm{Aut}(A)$. We write d_A^{-1} for $\xi\alpha_A$ and $\xi_*(d)$ for $\xi d\alpha_A d_A$ for each $d \in GA$; and we require that there be for each d a map $c: A \to A$ such that $\xi d = c\xi$; and that if $c \in \xi_* GA$ then $c\xi = \xi c(c)$ and $\xi F(c) = \xi$, and $cf = f$ implies $c = 1_A$ for all maps f with codomain A. A homomorphism $f: (A, \xi) \to (B, \zeta)$ is a map $f: A \to B$ such that $\zeta F(f) = f\xi$. The category of $\mathcal{Q}$-algebras and homomorphisms is denoted by $\mathcal{A}^{\mathcal{Q}}$.

Clearly a quintuple arises from any semiadjoint situation if we put $F = UV$, $G = EV$, $\mu = U\beta V$. Conversely, we have:

PROPOSITION 1. *Define* $U: \mathcal{A}^{\mathcal{Q}} \to \mathcal{A}$ *by* $U(A, \xi) = A$, $U(f) = f$, *and* $V: \mathcal{A} \to \mathcal{A}^{\mathcal{Q}}$ *by* $VA = (FA, \mu_A)$, $V(f) = F(f)$. *Then* V *is left semiadjoint to* U.

PROOF. We first begin to construct the functor E. Put $E(A, \xi) = \xi_* GA$ and $E(\xi) = \xi_*$. Then $\xi_*(d)\xi = \xi d$, for $d \in GA$, since $\xi d\alpha_A d_A \xi = c\xi\alpha_A d_A \xi = c\xi = \xi d$. Furthermore, $d_A = \xi_*(d_{VA} c(\xi)^{-1})$, since if d_A is defined by this equation, we have

$$\begin{aligned}\xi\alpha_A d_A \xi &= \xi\alpha_A \xi d_{VA} c(\xi)^{-1} = \xi c(\xi) F(\xi)\alpha_{FA} d_{VA} c(\xi)^{-1}\\ &= \xi_* c(\xi)\xi F(\xi)\alpha_{FA} d_{VA} c(\xi)^{-1} = \xi\xi\mu_A \alpha_{FA} d_{VA} c(\xi)^{-1}\\ &= \xi_* c(\xi)\xi c(\xi)^{-1} = \xi = 1_A \xi,\end{aligned}$$

so $\xi\alpha_A d_A = 1_A$ since ξ is epic. Thus, $d_A \in E(A, \xi)$ as required.

Now, let (B, ζ) be an algebra and $f: A \to B$. There exist maps $g: VA \to (B, \zeta)$ and $c \in E(B, \zeta)$ such that $f = cg\alpha_A$, namely $g = \zeta F(f)$ and $c = d_B\zeta_* c(f)$. We must show that c and g are unique; it is this step which requires the extra conditions imposed on ζ in the definition. Suppose $cg\alpha_A = dh\alpha_A$ where $h: VA \to (B, \zeta)$ and $d \in E(B, \zeta)$. By multiplying both sides by d^{-1} we may write the equation in the form $cg\alpha_A = h\alpha_A$; then

$$\begin{aligned} h &= h\mu_A F(\alpha_A) = \zeta F(h)F(\alpha_A) = \zeta F(cg\alpha_A) \\ &= \zeta F(c)F(g\alpha_A) = \zeta F(g)F(\alpha_A) \\ &= g\mu_A F(\alpha_A) = g, \end{aligned}$$

so $cg\alpha_A = g\alpha_A$ and $c = 1_B$.

Next let $f: (A, \xi) \to (B, \zeta)$ be a homomorphism. Since E is to be a functor, the map f_* will have to satisfy $\zeta_* G(f) = \zeta_* F(f)_* = f_*\xi_*$, and since ζ_* is epic this equation will define f_* provided it defines it consistently, i.e., provided $\zeta_*(c) = \zeta_*(d)$ implies $\zeta_* F(f)_*(c) = \zeta_* F(f)_*(d)$. But $\xi c\alpha_A d_A = \xi_*(c) = \xi_*(d) = \xi d\alpha_A d_A$ so $\xi c\alpha_A = \xi d\alpha_A$

$$\begin{aligned} f\xi c\alpha_A &= f\xi d\alpha_A \\ \zeta F(f)c\alpha_A &= \zeta F(f)d\alpha_A \\ \zeta_* F(f)_*(c)\zeta F(f)\alpha_A &= \zeta_* F(f)_*(d)\zeta F(f)\alpha_A \end{aligned}$$

and therefore $\zeta_* F(f)_*(c) = \zeta_* F(f)_*(d)$. Next, write $c = \xi_*(d)$; we must verify that $fc = f_*(c)f$. We have $f_*(c)f\xi = f_*\xi_*(d)f\xi = \zeta_* F(f)_*(d)\zeta F(f) = \zeta F(f)d = f\xi d = f\xi_*(d)\xi = fc\xi$ and the desired equation follows since ξ is epic. Now, ξ_* is a homomorphism since $\xi_*(cd)\xi = \xi cd = \xi_*(c)\xi d = \xi_*(c)\xi_*(d)\xi$ and ξ is epic; since ζ_*, $F(f)_* = G(f)$, and ξ_* are all homomorphisms, so is f_*. It is straightforward that $(gf)_* = g_* f_*$. Thus $E: \mathfrak{A}^Q \to \mathrm{Ab}$ is a functor. ∎

2. **Products.** The preservation properties, etc., of ordinary adjunctions do not carry over easily to the case of semiadjunctions. However, we can prove:

PROPOSITION 2. *Let $\mathfrak{A}$ be a category with finite products and $Q = (F, \alpha, \mu, G, c)$ a quintuple on $\mathfrak{A}$ with the property that $c_A \in E(A, \xi)$, $c_B \in E(B, \zeta)$, $p_A: A \times B \to A$ implies $G(p_A)(c(c_A \times c_B)) = c(c_A)$. Then $\mathfrak{A}^Q$ has finite products which are preserved by U.*

PROOF. We define $(A, \xi) \times (B, \zeta) = (A \times B, \rho)$ where $\rho = (\xi \times \zeta) \cdot (Fp_A, Fp_B): F(A \times B) \to A \times B$. There are six conditions on ρ to be verified. One sees straightforwardly that $\rho\alpha_{A\times B} = (\xi_*(c(p_A))d_A^{-1})$

$\times(\zeta_*(c(p_B))d_B^{-1})\in\mathrm{Aut}(A\times B)$. The condition $\rho F(\rho)=\rho\mu_{A\times B}$ follows as in the case of ordinary triples. If $d\in G(A\times B)$, it is easily seen that $\rho_*(d)=(\xi Fp_A)_*(d)\times(\zeta Fp_B)_*(d)$; thus, it follows that $E(A\times B,\ \rho)=E(A,\xi)\times E(B,\zeta)$. Then $p_A\rho F(c_A\times c_B)=\xi F(c_A)F(p_A)=\xi F(p_A)=p_A\rho$ and similarly $p_B\rho F(c_A\times c_B)=p_B\rho$ so $\rho F(c_A\times c_B)=\rho$. Suppose $f=(g,\ h):C\to A\times B$ and $(c_A\times c_B)f=f$. Then $(g,\ h)=(c_A\times c_B)(g,\ h)=(c_Ag,\ c_Bh)$ so $c_A=1_A$, $c_B=1_B$. Only the remaining condition requires our hypothesis on $G(p_A)$, namely that $(c_A\times c_B)\rho=\rho c(c_A\times c_B)$. For this, we have $p_A(c_A\times c_B)=p_A(\xi\times\zeta)(c(c_A)\times c(c_B))(Fp_A,\ Fp_B)=\xi c(c_A)F(p_A)=\xi F(p_A)c(c_A\times c_B)=p_A\rho c(c_A\times c_B)$ and similarly for p_B. Hence $(A\times B,\rho)$ is an object of $\mathcal{A}^{\mathbf{Q}}$ and that it serves as the product follows as in the case of triples. ∎

3. **Abelian groups in categories.** The standard example of a semiadjoint situation (see [1]) has $\mathcal{A}$ = category of abstract varieties and everywhere-defined rational maps, $\mathcal{B}$ = category of Abelian varieties, UB = the underlying variety of B, EB = the underlying group of B, VA = the Albanese variety of A. We will use for $\mathcal{A}$ instead the category of complete varieties and approach axiomatically the question of whether $\mathcal{B}\cong\mathcal{A}^{\mathbf{Q}}$.

Thus, let $\mathcal{A}$ be a category with finite products and a terminal object T, and let $\mathcal{B}$ be the category of abelian group structures on the objects of $\mathcal{A}$. Suppose $U:\mathcal{B}\to\mathcal{A}$ has a left semiadjoint V and let $\mathbf{Q}$ be the induced quintuple. We will write $t_A:A\to T$ and $e_B:T\to B$ for the unique maps .The obvious functor $\mathcal{B}\to\mathcal{A}^{\mathbf{Q}}$ is faithful; we will show it is full and representative. Let (A,ξ) be any $\mathbf{Q}$-algebra.

LEMMA. $\xi_*((Fd_A)_*(c(\xi))d_{VA}^{-1}c(d_A))=1_A$.

PROOF. We know that $\xi_*(d_{VA})=d_A\xi_*(c(\xi))$ so the equation is equivalent to

$$\begin{aligned} d_A &= \xi_*((Fd_A)_*(c(\xi))c(\xi)^{-1}c(d_A)) \\ &= d_A\xi_*(Fd_A)_*c(\xi)\xi_*c(\xi)^{-1} \end{aligned}$$

or $\xi_*(c(\xi))=\xi_*(Fd_A)_*c(\xi)$. This is equivalent to $\xi c(\xi)=\xi F(d_A)c(\xi)$ which is true since $\xi=\xi F(d_A)$. ∎

PROPOSITION 3. *$\mathcal{B}$ is equivalent to $\mathcal{A}^{\mathbf{Q}}$; in particular, the category of abelian varieties is "quintuplable" over the category of complete abstract varieties.*

PROOF. First we show that there is an abelian group structure on A making $\xi:FA\to A$ a homomorphism. Clearly since ξ is a retraction there is at most one such structure. Write $m_{FA}:FA\times FA\to FA$ for the multiplication, $i_{FA}: A \to A$ for the inverse and $e_{FA}: T \to FA$ for the identity. Define $m_A = \xi m_{FA}(\alpha_A d_A \times \alpha_A d_A)$, $i_A = \xi i_{FA} \alpha_A d_A$, $e_A = \xi e_{FA}$. The verifications of the group laws on A, of the fact that ξ is a homomorphism, and that any homomorphism $f: (A, \xi) \to (B, \zeta)$ is a homomorphism on the induced group structures are all variations of the same technique, so we will prove only the identity $m_A(i_A, 1_A) = e_A t_A$. Recall that m_{FA} is a homomorphism. Then:

$$
\begin{aligned}
m_A(i_A, 1_A)(\xi &= \xi m_{FA}(\alpha_A d_A \xi i_{FA} \alpha_A d_A \xi, \alpha_A d_A \xi) \\
&= \xi m_{FA}(F(\xi) k i_{FA} F(\xi) k, F(\xi) k) \\
&= \xi F(\xi) m_{FFA}(k,k)(i_{FA} F(\xi) k, 1_{FA}) \\
&= \xi \mu_A m_{FFA}(k,k)(i_{FA} F(\xi) k, 1_{FA}) \\
&= \xi m_{FA}(\mu_A k, \mu_A k)(i_{FA} F(\xi) k, 1_{FA}) \\
&= \xi m_{FA}(c,c)(i_{FA} F(\xi) k, 1_{FA}) \\
&= \xi_*(c^2) \xi m_{FA}(i_{FA}, 1_{FA})(F(\xi) k, 1_{FA}) \\
&= 1_A^2 \xi e_{FA} t_{FA}(F(\xi) k, 1_{FA}) \\
&= \xi e_{FA} t_{FA} = e_A t_{FA} = e_A t_A(\xi)
\end{aligned}
$$

where $k = (F(\alpha_A d_A))_* c(\xi) \alpha_{FA} c(d_A)$ and $c = (F d_A)_*(c(\xi)) d_{VA}^{-1} c(d_A)$, and we have applied the lemma. Now ξ is epic since it is a retraction, so we get $m_A(i_A, 1_A) = e_A t_A$ as required. ■

Reference

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