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ALMOST PERIODICITY OF THE INVERSE OF A FUNDAMENTAL MATRIX[1]

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ABSTRACT. We show that if X is the fundamental solution to $X' = AX + XB$ with X, A, and B almost periodic $n \times n$ matrices, then X^{-1} is almost periodic.

We prove the

THEOREM. *If X is the fundamental solution to $X' = AX + XB$ with X, A, and B almost periodic $n \times n$ matrices, then X^{-1} is almost periodic.*

This is apparently new even if A or B is the zero matrix. The equation is of extreme importance in the theory of stability since the transformation $Y = XZ$ maps the equation $Y' = AY$ into $Z' = -BZ$. It is useful to known that X^{-1} is bounded if X is. In particular, when X is almost periodic, then our theorem shows that X^{-1} is bounded. See Lillo [4] for applications. The fundamental solution is the solution such that $X(0) = I$.

The general result may be reduced to the scalar case in the following way. Since $X^{-1} = (\det X)^{-1}(\text{adjoint } X)^T$ it is easily seen that X^{-1} is almost periodic if and only if $(\det X)^{-1}$ is almost periodic. Now it is well known that if $Y' = AY$, $Y(0) = I$ then $y = \det(Y)$ satisfies $y' = (\text{trace } A)y$, $y(0) = 1$. Similarly if $Z' = ZB$, $Z(0) = I$, then $Z = \det(Z)$ satisfies $z' = (\text{trace } B)z$, $z(0) = 1$. Now it is easy to check that $X = YZ$ satisfies $X' = AX + XB$, $X(0) = I$ and that $x = \det(X)$ satisfies $x' = [\text{trace}(A + B)]x$, $x(0) = 1$. Consequently the theorem follows from the following lemma whose proof is contained in Bochner [1].

LEMMA. *Let y be a nontrivial scalar almost periodic solution to the almost periodic equation $y' = p(t)y$. Then $\inf|y| > 0$ and y^{-1} is almost periodic.*

If the result does not hold, let $\lim_n y(t_n) = 0$. By the almost periodicity of p and y we can find a subsequence s_n of t_n such that

$$\lim_n y(t + s_n) = z(t) \quad \text{and} \quad \lim_n p(t + s_n) = q(t)$$

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exist uniformly on the real line. Then $z' = qz$ and $z(0) = 0$. Hence $z(t) = 0$ for all t and $y(t) = \lim_n z(t - s_n) = 0$ for all t. So it must be that $\inf|y| > 0$. Hence y^{-1} is almost periodic.

It follows that $\int_0^t \operatorname{trace}(A+B)$ must be almost periodic if the fundamental solution of $X' = AX + XB$ is to be almost periodic in the real case. If A and B are complex, then

$$\int_0^t \operatorname{trace}(A + B) = iat + \text{an almost periodic function}, \quad \text{with } a \text{ real}.$$

This necessary condition, which seems to be new, has appeared as one of several sufficient conditions for the existence of an almost periodic vector solution, see e.g. [2]. See also Langenhop [3] for a related result if B is constant and X and X^{-1} are only required to be continuous and bounded.

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