ALMOST\textsuperscript{\textdagger}PERIODICITY OF THE INVERSE OF A FUNDAMENTAL MATRIX\textsuperscript{1}

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Abstract. We show that if $X$ is the fundamental solution to $X' = AX + XB$ with $X$, $A$, and $B$ almost periodic $n \times n$ matrices, then $X^{-1}$ is almost periodic.

We prove the

Theorem. If $X$ is the fundamental solution to $X' = AX + XB$ with $X$, $A$, and $B$ almost periodic $n \times n$ matrices, then $X^{-1}$ is almost periodic.

This is apparently new even if $A$ or $B$ is the zero matrix. The equation is of extreme importance in the theory of stability since the transformation $Y = XZ$ maps the equation $Y' = AY$ into $Z' = -BZ$.

It is useful to know that $X^{-1}$ is bounded if $X$ is. In particular, when $X$ is almost periodic, then our theorem shows that $X^{-1}$ is bounded. See Lillo [4] for applications. The fundamental solution is the solution such that $X(0) = I$.

The general result may be reduced to the scalar case in the following way. Since $X^{-1} = (\det X)^{-1} (\text{adjoint} X)^T$ it is easily seen that $X^{-1}$ is almost periodic if and only if $(\det X)^{-1}$ is almost periodic. Now it is well known that if $Y' = AY$, $Y(0) = I$ then $y = \det(Y)$ satisfies $y' = (\text{trace} A)y$, $y(0) = 1$. Similarly if $Z' = ZB$, $Z(0) = I$, then $z = \det(Z)$ satisfies $z' = (\text{trace} B)z$, $z(0) = 1$. Now it is easy to check that $X = YZ$ satisfies $X' = AX + XB$, $X(0) = I$ and that $x = \det(X)$ satisfies $x' = [\text{trace}(A + B)]x$, $x(0) = 1$. Consequently the theorem follows from the following lemma whose proof is contained in Bochner [1].

Lemma. Let $y$ be a nontrivial scalar almost periodic solution to the almost periodic equation $y' = p(t)y$. Then $\inf |y| > 0$ and $y^{-1}$ is almost periodic.

If the result does not hold, let $\lim_{n} y(t_{n}) = 0$. By the almost periodicity of $p$ and $y$ we can find a subsequence $s_{n}$ of $t_{n}$ such that

$$\lim_{n} y(t + s_{n}) = z(t) \quad \text{and} \quad \lim_{n} p(t + s_{n}) = q(t)$$

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exist uniformly on the real line. Then \( z'=qz \) and \( z(0)=0 \). Hence \( z(t)=0 \) for all \( t \) and \( y(t)=\lim_n z(t-s_n)=0 \) for all \( t \). So it must be that \( \inf|y|>0 \). Hence \( y^{-1} \) is almost periodic.

It follows that \( \int_0^t \text{trace}(A+B) \) must be almost periodic if the fundamental solution of \( X'=AX+XB \) is to be almost periodic in the real case. If \( A \) and \( B \) are complex, then

\[
\int_0^t \text{trace}(A+B) = i\alpha t + \text{an almost periodic function, \ with \ } a \text{ real.}
\]

This necessary condition, which seems to be new, has appeared as one of several sufficient conditions for the existence of an almost periodic vector solution, see e.g. [2]. See also Langenhop [3] for a related result if \( B \) is constant and \( X \) and \( X^{-1} \) are only required to be continuous and bounded.

**Bibliography**


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