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METRIZABILITY OF LOCALLY COMPACT VECTOR SPACES

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ABSTRACT. By use of the theory of characters and the Pontryagin-van Kampen theorem, it is shown that if E is a locally compact vector space over a discrete division ring K of characteristic zero and if $\dim_K E < 2^{\mathfrak{m}}$, where $\mathfrak{m}$ is the cardinality of K, then E is metrizable.

The problem of determining whether a locally compact vector space over a discrete division ring is metrizable arises in the study of finite-dimensional locally compact vector spaces, because we have a fairly concrete picture of those that are metrizable: If E is a finite-dimensional, metrizable, indiscrete locally compact vector space over a discrete field K and if $\mathfrak{o}$ is the smallest open subspace of E, then the topological additive group $\mathfrak{o}$ admits the structure of finite-dimensional topological vector space over the locally compact field F, where F is either the real field $\boldsymbol{R}$, the field $\boldsymbol{Q}_q$ of q-adic numbers, or the field $\boldsymbol{Z}_p((X))$ of power series over the field $\boldsymbol{Z}_p$ of integers modulo p, under a scalar multiplication satisfying $\alpha.(\lambda x) = \lambda(\alpha.x)$ for all $x \in E$, $\lambda \in K$, $\alpha \in F$; moreover, if E is a topological algebra, then $\mathfrak{o}$ is an ideal and $\alpha.(xy) = (\alpha.x)y$, $\alpha.(yx) = y(\alpha.x)$ for all $\alpha \in F$, $x \in \mathfrak{o}$, $y \in A$; finally, K is algebraically isomorphic to a subfield of finite codegree of a finite extension of F [**4**, Theorems 3 and 5]. Here we shall consider the special case of this problem where the scalar field has characteristic zero.

First, we need a lower bound on the dimension of nonzero compact vector spaces. Let K be a division ring, equipped with the discrete topology. We denote by $K^\wedge$ the (compact) character group of the discrete additive group K, made into a right topological vector space over K by defining $u.\lambda: x \mapsto u(\lambda x)$ for all $u \in K^\wedge$, $\lambda \in K$, $x \in K$ [**3**, Theorem 1].

THEOREM 1. *If K is an infinite division ring of cardinality $\mathfrak{m}$, then* $\dim_K K^\wedge = 2^{\mathfrak{m}}$.

PROOF. *Case* 1. The characteristic of K is zero. Then for some cardinal number $\mathfrak{n}$ the additive group of K is isomorphic to $\boldsymbol{Q}^{(\mathfrak{n})}$, the

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direct sum of $\mathfrak{n}$ copies of the additive group $\boldsymbol{Q}$ of rationals, where $\mathfrak{n}=\mathfrak{m}$ if $\mathfrak{m}>\aleph_0$ and where $1\leqq\mathfrak{n}\leqq\aleph_0$ if $\mathfrak{m}=\aleph_0$. Hence $K^\wedge$ is topologically isomorphic to $(\boldsymbol{Q}^\wedge)^{\mathfrak{n}}$, the cartesian product of $\mathfrak{n}$ copies of $\boldsymbol{Q}^\wedge$ [2, (23.21), p. 364], and card $(\boldsymbol{Q}^\wedge)=\mathfrak{c}$ [2, (25.4), p. 404]. If $\mathfrak{m}>\aleph_0$, then card $(K^\wedge)=\mathfrak{c}^{\mathfrak{n}}=\mathfrak{c}^{\mathfrak{m}}=2^{\mathfrak{m}}>\mathfrak{m}$, so $\dim_K K^\wedge=2^{\mathfrak{m}}$. If $\mathfrak{m}=\aleph_0$, then card$(K^\wedge)$ $=\mathfrak{c}^{\mathfrak{n}}=\mathfrak{c}>\mathfrak{m}$, whence again $\dim_K K^\wedge=2^{\mathfrak{m}}$.

Case 2. The characteristic of K is a prime p. Then the additive group of K is isomorphic to $\boldsymbol{Z}_p^{(\mathfrak{m})}$, so $K^\wedge$ is topologically isomorphic to $(\boldsymbol{Z}_p)^{\mathfrak{m}}$. Hence card$(K^\wedge)=p^{\mathfrak{m}}=2^{\mathfrak{m}}>\mathfrak{m}$, so $\dim_K K^\wedge=2^{\mathfrak{m}}$.

As a consequence of Theorem 1, we note that if K is uncountable, then $K^\wedge$ is a nonmetrizable compact K-vector space of dimension $2^{\mathfrak{m}}$ [3, Theorem 8].

THEOREM 2. *Let K be a discrete division ring of characteristic zero. If E is a locally compact, totally disconnected K-vector space, then E is metrizable.*

PROOF. Let Q be the prime field of K. By [2, (7.7), p. 62], E contains a compact open subgroup V. Let $F=\bigcap\{\alpha V:\alpha\in Q^*\}$. Then F is a compact vector space over Q and hence is connected [3, Theorem 9]. Thus $F=(0)$. Hence, as V is compact, for any neighborhood W of zero there exist $\alpha_1,\ldots,\alpha_n\in Q^*$ such that $W\supseteq\alpha_1V\cap\ldots\cap\alpha_nV$. Therefore $\{\alpha_1V\cap\ldots\cap\alpha_nV:\alpha_1,\ldots,\alpha_n\in Q^*\}$ is a fundamental system of neighborhoods of zero in E; in particular, E is metrizable.

THEOREM 3. *Let K be a discrete division ring of characteristic zero, and let $\mathfrak{m}=\mathrm{card}(K)$. If E is a locally compact K-vector space and if $\dim_K E<2^{\mathfrak{m}}$, then E is metrizable.*

PROOF. Let C be the connected component of zero. By Theorem 2, E/C is metrizable. By [2, (e), p. 47], it therefore suffices to show that C is metrizable. Hence we may assume that E is connected. By the theorem of Pontryagin and van Kampen [2, (9.14), p. 95], the topological additive group E is the topological direct sum of $\boldsymbol{R}^m$ and H, where H is a compact subgroup. Let u be the (continuous) projection of E on $\boldsymbol{R}^m$ along H. If $h\in H$, then the closed additive subgroup $(\boldsymbol{Z}h)^-$ generated by h is compact as it is contained in H; if $\lambda\in K$, then $(\boldsymbol{Z}\lambda h)^-=\lambda(\boldsymbol{Z}h)^-$, a compact subgroup, whence $u((\boldsymbol{Z}\lambda h)^-)=(0)$ as $\boldsymbol{R}^m$ contains no nonzero compact additive subgroups, and therefore $\lambda h\in(\boldsymbol{Z}\lambda h)^-\subseteq H$. Hence H is a vector subspace of E. By Theorem 1, [3, Theorem 6], and our hypothesis, $H=(0)$. Hence $E=\boldsymbol{R}^m$ and thus is metrizable.

If K is countable, we may improve Theorem 3:

THEOREM 4. *Assume the Continuum Hypothesis. If K is a countable, discrete division ring of characteristic zero and if E is a locally compact*

K-vector space such that $\dim_K E \leqq \mathfrak{c}$, *then E is metrizable.*

PROOF. As in the proof of Theorem 3, we may assume that E is the topological direct sum of $\boldsymbol{R}^m$ and a compact subspace H. By [3, Theorem 6], H is topologically isomorphic to the compact K-vector space $(K^\wedge)^\mathfrak{n}$, the cartesian product of $\mathfrak{n}$ copies of $K^\wedge$, for some cardinal number $\mathfrak{n}$. If $\mathfrak{n} > \aleph_0$, then $\operatorname{card}(K^\wedge)^\mathfrak{n} = \mathfrak{c}^\mathfrak{n} > \mathfrak{c}$, so $\dim_K E \geqq \dim_K H > \mathfrak{c}$, a contradiction. Hence $\mathfrak{n} \leqq \aleph_0$, so H is metrizable as $K^\wedge$ is [3, Theorem 8]. Thus E is metrizable.

It is an open question whether similar theorems hold for locally compact vector spaces over fields of prime characteristic. At any rate, we may take care of the one-dimensional case:

THEOREM 5. *If E is an indiscrete, one-dimensional locally compact vector space over a discrete field K, then there is a topology on K making K into an indiscrete locally compact field and E a topological vector space over K, so topologized; in particular, E is metrizable.*

PROOF. The proof is similar to that of [3, Theorem 10]. We topologize K so that $f: \lambda \mapsto \lambda a$ is a homeomorphism, where a is a nonzero vector. Then K is locally compact; $(\lambda, \mu) \mapsto \lambda + \mu$ is continuous, since each of the maps $(\lambda, \mu) \mapsto (\lambda a, \mu a) \mapsto \lambda a + \mu a = (\lambda + \mu)a \mapsto \lambda + \mu$ is; and for each $\alpha \in K$, $\lambda \mapsto \alpha\lambda$ is continuous, since each of the maps $\lambda \mapsto \lambda a \mapsto \alpha\lambda a \mapsto \alpha\lambda$ is. With the induced topology, the multiplicative group K^* satisfies the hypotheses of Ellis's theorem [1, Theorem 2], so K^* is a locally compact group. In particular, the mapping $(\lambda, \mu) \mapsto \lambda\mu$ is continuous at $(1, 1)$; it is therefore also continuous at $(0,0)$, for if V is a neighborhood of zero, there exists a neighborhood U of zero such that $(1+U)(1+U) \subseteq 1 + V$, whence $UU \subseteq U + U + UU = (1+U)(1+U) - 1 \subseteq V$. Therefore K is an indiscrete locally compact field, so its topology is given by an absolute value; consequently, E is also metrizable. Clearly E is a topological vector space over K, as each of the maps $(\lambda, \mu a) \mapsto (\lambda, \mu) \mapsto \lambda\mu \mapsto \lambda\mu a$ is continuous.

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