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# A SPECIAL BASIS FOR $C([0, 1])$

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ABSTRACT. This paper constructs a basis for $C([0, 1])$ which converges weakly to zero whose elements are nevertheless norm bounded away from zero.

**Introduction.** A basis for a Banach space $X$ is a sequence $\{x_n\} \subset X$ such that to each $y \in X$ there exists a unique sequence of scalars $c_n$ for which the partial sums of $\sum c_n x_n$ converge to $y$ in the norm of $X$. Bases can be distinguished by combinatorial and topological criteria (cf. [4]). Foiaş and Singer asked in [1] if $C([0, 1])$ possesses a basis of type $wc_0$, that is, a basis that converges weakly to zero whose elements have norms bounded away from zero. This paper constructs such a basis using a scheme that may be useful in looking for other peculiar bases in $C([0, 1])$.

J. R. Holub has recently [2] constructed a basis in $c_0$ that is weakly closed yet intersects every weak neighborhood of zero defined by a single linear functional. The referee has pointed out that Holub's example and the construction below easily imply the existence of such a basis in $C([0, 1])$.

**A method of construction.** A sequence of piecewise linear functions which form a basis in $C([0, 1])$ can be constructed in the following way. The sequence of functions will be accompanied by a parallel sequence of points in $[0, 1]$. These points will be called joints. The joints are to mark the intervals on which the functions are actually linear. The functions and joints will be grouped into successive stages, each stage having some finite number of functions and an equal number of joints. The $k$th function (joint) of the $n$th stage will be denoted $f_{n,k}$ ($a_{n,k}$). To simplify notation $a_{n,k}$ will sometimes be written $a(n, k)$. To form a basis the $f_{n,k}$ must be linearly ordered. Do this by saying $f_{i,j}$ precedes $f_{n,k}$ if $i<n$ or if $i=n$ and $h<j$. Order the $a_{n,k}$ accordingly.

The joints $a_{n,k}$ will be defined first. Some notation is needed. Let $j_n$ be the number of joints in the $n$th stage. Let $J_{n,k}$ be the set of joints consisting of $a_{n,k}$ and all preceding joints. Put $J_n = J_{n,j_n}$. In visualizing the construction notice that the distance between any two adjacent points of $J_{n,k} \cup \{0, 1\}$ is $2^{-m}$ for some nonnegative integer $m$

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depending on the particular adjacent points. Put $f_{1,1}(x)\equiv 1$. Thereafter define $f_{n,k}$ to be that function which

(1) is continuous on $[0, 1]$;

(2) is linear between adjacent points of $J_n$;

(3) is linear between zero (one) and the second leftmost(rightmost) point of $J_n$ if $0\notin J_n$ $(I\notin J_n)$;

(4) takes a given value at each point of $J_n$.

In reference to (4), it will be true that $f_{n,k}$ vanishes on $J_{n-1}$; moreover a value $f_{n,k}(a_{i,j})$ will be given only if it is nonzero.

To illustrate the operation of this scheme we use it to describe Schauder's original basis for $C([0, 1])$; cf. [**3**, p. 49]. Each stage has one function and one joint, and we drop the second subscript. Put $a_1=0$, $a_2=1$ and $a(2^k+l+1)=2^{-k-1}+(l-1)2^{-k}$, for $0\leqq k$ and $1\leqq l \leqq 2^k$. The basis functions are defined by $f_n(a_n)=1$.

Let $\|f\| =\sup\{|f(x)| :0\leqq x\leqq 1\}$. A feature of Schauder's basis is that $\|f_n\|=1$ for all $n$. In the construction below it will be true that $\|f_{n,k}\|=1$ for all $n$ and $k$. In checking this and other bounds it is helpful to observe that

$$\|f_{n,k}\| = \max\{ |f_{n,k}(x)| :x\in \{0, 1\} \cup J_n\}.$$

Once a sequence $\{f_{n,k}\}$ is defined it must be shown to be a basis. Suppose that $g\in C([0, 1])$ and that $\sum c_{n,k}f_{n,k}$ (summed according to the ordering of the $f_{n,k}$) is a candidate for the basis representation of $g$. Put $P_{n,k}=\sum_{i=1}^{n}\sum_{j=1}^{k} c_{i,j}f_{i,j}$; set $P_n=P_{n,j_n}$ and $g_n=g-P_n$, $g_0=g$. Since for $m>n$ the functions $f_{m,j}$ all vanish on $J_n$, $P_n$ must interpolate $g$ on $J_n$. To see that this is possible it is enough to check that for each $n$ the restrictions of the functions in the $n$th stage to the set of joints in the $n$th stage are linearly independent. Indeed, if $B_n$ is the linear transformation whose matrix is $(b_{i,j})=(f_{n,j}(a_{n,i}))$, then $(c_{n,j}) =B_n^{-1}(g_{n-1}(a_{n,i}))$. This also shows that the $c_{n,k}$ must be unique, if the basis representation always does exist. In any event there is a unique sequence $\{c_{n,k}\}$ such that for each $n$, $P_n$ interpolates $g$ on $J_n$.

Let $\mu_n$, also written $\mu(n)$, be the mesh of $J_n\cup\{0, 1\}$ and let $w(t)$ be the modulus of continuity of $g$. Between any two adjacent points of $J_n$ the functions $P_n$ and $g$ differ by at most $w(\mu_n)$. This is an easy consequence of the monotonicity of $w$ and the well-known result:

LEMMA 1. *If $g$ is continuous on $[a, b]$ with modulus of continuity $w$, and $P$ is the linear function interpolating $g$ at $a$ and $b$, then $a\leqq x\leqq b$ implies $|g(x)-P(x)|\leqq w(b-a)$.*

PROOF. For $a\leqq x\leqq b$ we have $|g(x)-g(a)|\leqq w(x-a)$ and $|g(x) -g(b)|\leqq w(b-x)$. Since $w$ is nondecreasing, $w(x-a)$ and $w(b-x)$ are less than or equal to $w(b-a)$. Thus

$$\begin{aligned} |g(x) - P(x)| &= |(b - x)(g(x) - g(a)) \\ &\qquad + (x - a)(g(x) - g(b))|/(b - a) \\ &\leqq [(b - x)w(x - a) + (x - a)w(b - x)]/(b - a) \\ &\leqq w(b - a). \end{aligned}$$

The relation between $P_n$ and $g$ near zero (one) if zero (one) is not in $J_n$ is more delicate. In that case the sequence $\{a_{n,k}\}$ should be so chosen that if $0 \notin J_n$ $(1 \notin J_n)$ then the distance from zero (one) to the leftmost (rightmost) point of $J_n$ is exactly half the distance from zero (one) to the second leftmost (rightmost) point. We use the following easy result:

LEMMA 2. *Let $g$ be continuous on $[0, 1]$ with modulus of continuity $w$. Fix $t \in [0, \frac{1}{2}]$. If $P$ is the linear function which interpolates $g$ at the points $t$ and $2t$ $(1-2t$ and $1-t)$, then for $0 \leqq x \leqq t$ $(1-t \leqq x \leqq 1)$ we have $|g(x)-P(x)| \leqq 2w(t)$.*

PROOF. If $P$ interpolates $g$ at $t$ and $2t$ then, for $0 \leqq x \leqq t$,

$$\begin{aligned} |g(x) - P(x)| &= |g(x) - g(t) - (x - t)(g(2t) - g(t))/t| \\ &\leqq w(t) + |x - t| w(t)/t \leqq 2w(t). \end{aligned}$$

The case of interpolation at $1-2t$ and $1-t$ is proved similarly.

From Lemmas 1 and 2 and the chosen spacing of the joints $a_{n,k}$ we will have in each construction that $\|g-P_n\| \leqq 2w(\mu_n)$. For the $P_n$ to converge uniformly to $g$ it suffices that $\mu_n \to 0$. In fact, since $\mu_{n+1} \leqq \mu_n$, it is enough to check that some subsequence of the $\mu_n$ tends to zero.

One notes that the sequence $\{P_n\}$ is not the full sequence of partial sums of $\sum c_{n,k}f_{n,k}$. To show that the full sequence of partial sums converges uniformly to $g$ we will find a finite constant $M$ such that for all $n$ and $1 \leqq k \leqq j_n$ we have $\|P_{n,k}-P_{n-1}\| \leqq M\|g-P_{n-1}\|$. When this is done we will have $\|g-P_{n,k}\| \leqq (1+M)\|g-P_{n-1}\|$ and we will already know that $\|g-P_{n-1}\|$ goes to zero.

**The $wc_0$ basis.** In the first stage there are three joints: $a_{1,1}=0$, $a_{1,2}=\frac{1}{2}$, $a_{1,3}=1$. As always $f_{1,1}(x) \equiv 1$; the other two functions in the first stage are defined by $f_{1,2}(a_{1,2}) = f_{1,3}(a_{1,3}) = 1$. The definition of succeeding stages differs as $n$ is even or odd. The $2n$th stage has $2^n$ joints: $a_{2n,1} = 1-2^{-n-1}$ and $a(2n,\ 2^k+l) = 1-2^{-n}+2^{-n-k-1}(l-\frac{1}{2})$, $0 \leqq k \leqq n-1$, $1 \leqq l \leqq 2^k$. For simplicity we momentarily depart from the usual scheme of defining the corresponding functions and put

$$f_{2n,k}(x) = f_{k+1}(2^{n+1}(x - 1 + 2^{-n}))$$

for $1-2^{-n} \leqq x \leqq 1-2^{-n-1}$ and $1 \leqq k \leqq 2^n$. Here $\{f_k\}$ is Schauder's basis as defined above in the introduction. On the balance of $[0, 1]$ $f_{2n,k}$ is defined as usual—namely, it is continuous, linear between adjacent points of $J_{2n}$ and zero at joints in $J_{2n}$ where it is not already defined.

The $(2n+1)$th stage has $2^n+1$ joints. For $2 \leqq l \leqq 2^n+1$ put $a_{2n+1,l} = 1-2^{-n}+2^{-2n-2}+(l-2)2^{-2n-1}$. The joints $a_{2n+1,1}$ will be defined inductively along with two sequences of natural numbers $\{n_k\}$ and $\{m_k\}$. Let $a_{3,1}=2^{-2}$, $n_1=1$ and $m_1=1$. Suppose $n_k$, $m_k$ and $a(2n_k+1, 1)$ have been chosen. For $1 \leqq l \leqq m_k-1$ put $a(2(n_k+l)+1, 1)=2^{-k-1}+l2^{-k}$. Set $n_{k+1}=n_k+m_k$ and $a(2n_{k+1}+1, 1)=2^{-k-2}$. Let $m_{k+1}$ be the number of successive intervals to the right of zero which have length $2^{-k-1}$ and lie between adjacent points of $J(2n_{k+1})$. For example $n_2=m_2=2$ and $n_4=8$, $m_4=12$. Now let $f_{2n+1,k}(a_{2n+1,1})=(2^n+1)^{-1}$ for $1 \leqq k \leqq 2^n+1$. For $2 \leqq k \leqq 2^n+1$ let $f_{2n+1,1}(a_{2n+1,k}) = -f_{2n+1,k}(a_{2n+1,k})=1$.

The linear independence condition in even stages is easily checked by a comparison with Schauder's basis; in odd stages it is a consequence of the solutions for the $c_{n,k}$ given below. We have $\mu(2n_k)=2^{-k}$, so the mesh condition holds. In even stages we note that $P_{2n,k}$ interpolates $g$ on $J_{2n,k}$. Lemma 1 and other remarks in the introduction show that $\|g-P_{2n,k}\| \leqq w(g;\, \mu_{2n-1})$. Consider $P_{2n+1,k}$. It interpolates $g$ on $J_{2n}$.

We will prove that the $P_{2n+1,k}$ converge to $g$ by showing that the norm of $h_{n,k}=P_{2n+1,k}-P_{2n}$ is at most $3\|g-P_{2n}\|$. That is, in the terminology of the general construction we can choose $M=3$. Moreover, the norm of $h_{n,k}$ is $\max_l |h_{n,k}(a_{2n+1,l})|$ because $h_{n,k}$ is linear between adjacent points of $J_{2n+1}$ and vanishes off the joints of the $(2n+1)$th stage. It is therefore sufficient that $h_{n,k}$ satisfy the bound at the points $a_{2n+1,l}$.

Fix $n$ and set $a_n=\|g-P_{2n}\|$. Put $v_k=g(a_{2n+1,k})-P_{2n}(a_{2n+1,k})$ and write $c_k$ for $c_{2n+1,k}$. Write $m$ for $2^n+1$. Since $h_{n,k}$ interpolates $g-P_{2n}$ on the joints of the $n$th stage we have $(c_1+\cdots+c_m)/m=v_1$ and $c_1-c_k=v_k$, for $2 \leqq k \leqq m$. Solving for the $c_k$ we get $c_1=v_1+(1/m)\sum_{k=2}^m v_k$ and for $2 \leqq k \leqq m$, $c_k=c_1-v_k$. We see that $|c_1| \leqq |v_1|+(1/m)\sum_2^m |v_k| \leqq 2a_n$, and hence that $\|h_{n,1}\|=\|c_1 f_{2n+1,1}\| \leqq 2a_n$. For $2 \leqq k \leqq m$ we have

$$\begin{aligned} |h_{n,k}(a_{2n+1,1})| &= \left|(k/m)c_1 - (1/m)\sum_2^k v_l\right| \\ &\leqq (k/m)|c_1| + (k-1)a_n/m \leqq 3a_n. \end{aligned}$$

On $\{a(2n+1, 2), \cdots, a(2n+1, k)\}$ $h_{n,k}$ interpolates $g-P_{2n}$ and on $\{a(2n+1, k+1), \cdots, a(2n+1, m)\}$ $h_{n,k}$ agrees with $h_{n,1}$. It follows that $\|h_{n,k}\| \leqq 3a_n$. Since $a_n \to 0$ we are done showing that $\{f_{n,k}\}$ is a basis.

To see that $f_{n,k}$ goes weakly to zero, recall that weak convergence to zero is equivalent to boundedness and pointwise convergence to zero. We have $\|f_{n,k}\| \equiv 1$. On $[0, 1-2^{-n}]$ $f_{2n,k}$ vanishes and $f_{2n+1,k}$ has supremum $(2^n+1)^{-1}$. For $n>1$, $f_{n,k}(1)=0$. Thus we have pointwise convergence to zero on $[0, 1]$. Hence $\{f_{n,k}\}$ is $wc_0$.

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