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SHORTER NOTES

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SUBMANIFOLDS IN A EUCLIDEAN HYPERSPHERE

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ABSTRACT. Let M be an oriented closed n-dimensional submanifold of a euclidean $(n+N)$-space E^{n+N}. Let $\boldsymbol{X}$ and $\boldsymbol{H}$ be the position vector field and the mean curvature vector field of M in E^{n+N}. Then M is contained in a hypersphere of E^{n+N} centered at $\mathbf{c}$ when and only when either $(\boldsymbol{X}-\mathbf{c})\cdot\boldsymbol{H}\geqq -1$ or $(\boldsymbol{X}-\mathbf{c})\cdot\boldsymbol{H}\leqq -1$.

Let M be an oriented closed n-dimensional submanifold of a euclidean space E^{n+N} of dimension $n+N$, and let $\boldsymbol{X}$ be the position vector field of M in E^{n+N}. Then M is a riemannian manifold with the induced metric. In the following, let ∇ and ∇' be the covariant differentiations of M and E^{n+N}, respectively. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two tangent vector fields on M. Then the Gauss formula gives

$$\nabla'_{\boldsymbol{u}}\boldsymbol{v} = \nabla_{\boldsymbol{u}}\boldsymbol{v} + \alpha(\boldsymbol{u}, \boldsymbol{v}), \tag{1}$$

where $\alpha(\boldsymbol{u}, \boldsymbol{v})$ is the second fundamental form of M. If $\{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n\}$ is an orthonormal basis in the tangent space $T_x(M)$, then the normal vector

$$\boldsymbol{H} = (1/n)\sum_{i=1}^{n}\alpha(\boldsymbol{e}_i, \boldsymbol{e}_i) \tag{2}$$

is called the mean curvature normal at x. The main purpose of this note is to prove the following:

THEOREM 1. *Let M be an oriented closed n-dimensional submanifold of E^{n+N}. Then M is contained in a hypersphere of E^{n+N} centered at $\mathbf{c}\in E^{n+N}$ if and only if we have either $(\boldsymbol{X}-\mathbf{c})\cdot\boldsymbol{H}\geqq -1$ or $(\boldsymbol{X}-\mathbf{c})\cdot\boldsymbol{H}\leqq -1$.*

PROOF. Let $\mathbf{c}$ be a fixed vector in E^{n+N}. Then by a direct computation for the Laplacian Δ of the function f:

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$$f = (\boldsymbol{X} - \mathbf{c})\cdot(\boldsymbol{X} - \mathbf{c}) \tag{3}$$

on M, we get

$$\Delta f = 2n(1 + (\boldsymbol{X} - \mathbf{c})\cdot \boldsymbol{H}). \tag{4}$$

Therefore, we know that if we have either $(\boldsymbol{X}-\mathbf{c})\cdot\boldsymbol{H} \geqq -1$ or $(\boldsymbol{X}-\mathbf{c})\cdot\boldsymbol{H} \leqq -1$, then the function f is a constant. Thus M is contained in a hypersphere of E^{n+N} centered at $\mathbf{c}$. Conversely, if M is contained in a hypersphere of E^{n+N} centered at $\mathbf{c}$, then the function f is a constant. Hence, by formula (4), we get $(\boldsymbol{X}-\mathbf{c})\cdot\boldsymbol{H} = -1$. This completes the proof of the theorem.

If the codimension $N=1$, then the mean curvature H of M is given by $\boldsymbol{H} = H\boldsymbol{e}$, where $\boldsymbol{e}$ is the unit outer normal vector field, and the support function p is given by $\boldsymbol{X}\cdot\boldsymbol{e}$. From Theorem 1, we get

COROLLARY. *Let M be an oriented closed hypersurface of E^{n+1}. Suppose that either $pH \geqq -1$ or $pH \leqq -1$. Then M is a hypersphere.*

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