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COUNTEREXAMPLES ON THE RANK OF A MANIFOLD[1]

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ABSTRACT. For any given integer r a closed manifold is constructed which has a smooth free action of the r-torus, and hence has rank at least r, but for which -1 is not a multiple root of the Poincaré polynomial.

Let M^n be an orientable, closed, differentiable n-manifold. The rank of M is defined to be the maximum number of (everywhere) independent *commuting* vector fields on M. The Poincaré polynomial of M is $P_M(t)=\sum b_i t^i$ where b_i is the ith Betti number of M. Thus rank $M \geqq 1$ iff -1 is a root of $P_M(t)$. It has been conjectured (see [1, p. 669]) that if rank $M>1$ then -1 is a multiple root of $P_M(t)$. A weaker conjecture is that if the 2-torus T acts freely and smoothly on M (i.e., M is the total space of a principal T-bundle) then -1 is a multiple root of $P_M(t)$.

This conjecture is so attractive that perhaps no serious attempt has been made to *disprove* it. At least that is the only explanation for the simplicity of the counterexamples which we shall now give.

If $A^k \subset M^n$ is a k-dimensional closed submanifold of M, then we denote by $d(M, A)$ the n-manifold obtained by removing an open tubular neighborhood of A from M and doubling the remainder along its boundary. Let us compute the Poincaré polynomial of $d(M, A)$. We shall *assume* that

$$\tilde{H}_*(A) \to \tilde{H}_*(M) \quad \text{is trivial and} \quad n \geqq 2k+3$$

partially for convenience of computation, but mostly since the conclusions would be false without some such assumptions.

If M_0 is the complement of an open tube about A, then $M_0 \subset M$ induces an isomorphism in homology through the middle dimension, since dim $A=k \leqq (n-3)/2$. The boundary of M_0 is an $(n-k-1)$-sphere bundle over A and $k<n-k-1$, by assumption, so that $H_p(\partial M_0) \approx H_p(A)$ for $p \leqq n-k-2$. Since $n-k-2 \geqq [n/2]$, this holds through the middle dimension. The Mayer-Vietoris sequence

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$$\cdots \to H_p(\partial M_0) \xrightarrow{i_*} H_p(M_0) \oplus H_p(M_0) \xrightarrow{j_*} H_p(d(M, A))$$
$$\xrightarrow{\partial_*} H_{p-1}(\partial M_0) \to \cdots$$

has $i_* = 0$ (for $p \neq 0$) at least through the middle dimension, by the assumption that A is homologically trivial in M. Thus, through the middle dimension, $P_{d(M,A)}(t)$ coincides with $2P_M(t) - 1 + t(P_A(t) - 1)$. By Poincaré duality, we conclude that

$$\begin{aligned} P_{d(M,A)}(t) &= 2P_M(t) - 1 - t^n + t(P_A(t) - 1) + t^{n-k-1}(P_A(t) - t^k) \\ &= 2P_M(t) + (t + t^{n-k-1})P_A(t) - (1 + t)(1 + t^{n-1}). \end{aligned}$$

Now suppose that n is odd and that the r-torus T^r ($r \geqq 2$) acts freely on M with A invariant. Also suppose that $(1+t)^2$ divides both $P_M(t)$ and $P_A(t)$. (The simplest example is $r = 2$, $k = 2$, $M = S^3 \times S^3 \times S^1$ and $A = T^2$ an orbit of the obvious T^2 action on the first two factors of $S^3 \times S^3 \times S^1$.) Then T^r also acts freely on $d(M, A)$, so that $\operatorname{rank}(d(M, A)) \geqq r$, but

$$P_{d(M,A)}(t) \equiv -(1 + t)(1 + t^{n-1}) \not\equiv 0 \ \left[\operatorname{mod}\ (1 + t)^2\right],$$

(for n odd), so that -1 is a simple root of $P_{d(M,A)}(t)$. Note that, for example, we can take $M = S^m \times S^m \times \cdots \times S^m$ (an odd number $\geqq r \geqq 2$ of copies, and m odd) with $T^r = S^1 \times \cdots \times S^1$ acting in the standard way on the first r "coordinates," and A can be taken to be $S^p \times \cdots \times S^p \times * \times \cdots \times *$ (r copies of S^p and p odd) for m much larger than p. Then $d(M, A)$ is p-connected. Thus we have

THEOREM A. *There are manifolds of arbitrarily high rank and connectivity whose Poincaré polynomials have -1 as a simple root.*

For n even, the Poincaré polynomial cannot have -1 as a simple root, since if $P_M(t) = (1+t)Q(t)$, then $Q(t)$ obeys formal Poincaré duality and has odd degree, whence $Q(-1) = 0$. Note however that if $(1+t)^3$ divides both $P_M(t)$ and $P_A(t)$, then (for n even)

$$P_{d(M,A)}(t) \equiv -(1+t)(1+t^{n-1}) = -(1+t)^2(1-t+t^2- \cdots +t^{n-2}) \not\equiv 0 \quad \left[\operatorname{mod}\ (1+t)^3\right].$$

Taking an even number of copies of S^m in the above example, we then have

THEOREM B. *There are even dimensional manifolds of arbitrarily high rank and connectivity whose Poincaré polynomials have -1 as (only) a double root.*

These examples would seem to indicate that, despite intuition, the rank of a manifold and the multiplicity of -1 as a root of its Poincaré polynomial, have very little to do with each other.

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