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# HYPERGRAPHS AND RAMSEYIAN THEOREMS

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ABSTRACT. A $k$-graph is an ordered couple $(V, E)$ where $V$ is a set and $E$ a set of $k$-tuples of elements of $V$; thus, a 2-graph is an ordinary graph. If the notions of the independent set and the chromatic number are generalized for $k$-graphs then one can ask what is the least number of edges in a $k$-graph having $p$ vertices and no independent set of size $b$ (the problem of Turán) and what is the least number of edges in a $k$-graph whose chromatic number exceeds a given number (the generalized problem of Erdös and Hajnal). As in graphs, there is a relationship between independent sets and chromatic numbers—actually, our results for the first problem are applicable to the second one. The theorems of Ramsey's type are, in fact, theorems on the chromatic number of certain $k$-graphs; thus, the results for the problem of Erdös and Hajnal yield lower bounds for the general "Ramseyian numbers".

**1. Independent sets—a problem of Turán.** A *hypergraph* (set-system in [6]) is an ordered couple $(V, E)$ where $V = V(H)$ is a set (the set of *vertices* of $H$) and $E = E(H)$ is a set of some subsets of $V$ (the set of *edges* of $H$). We write $p(H) = |V|$, $q(H) = |E|$.

If $X$ is a set and $k$ a positive integer then we write

$$[X]^k = \{Y \subset X : |Y| = k\}.$$

A *$k$-graph* is a hypergraph $H$ such that $E(H) \subset [V(H)]^k$.

A set $B \subset V(H)$ is called *independent* (in $H$) if $A \subset B$ for no $A \in E(H)$. The maximum of the cardinalities of the independent sets in $H$ is denoted by $\beta(H)$.

If $p, k, b$ are positive integers, $k \leqq b \leqq p$, then we denote by $T(p, k, b)$ the smallest $q$ such that there exists a $k$-graph $H$ with properties $p(H) = p$, $q(H) = q$, $\beta(H) < b$.

Turán [12] determined $T(p, 2, b)$ and asked [13] for the values of $T(p, k, b)$ in general. We will assume $1 < k < b < p$ as the other cases are trivial.

Katona, Nemetz and Simonovits [9] have shown, inter alia, that

$$T(p, k, b) \geqq \binom{p}{k}\binom{b}{k}^{-1} \tag{1}$$

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and remarked that a construction due to Turán yields

$$T(p, k, b) \leqq \binom{p}{k}\left[\frac{b-1}{k-1}\right]^{1-k}. \tag{2}$$

Actually, (1) can be slightly improved into

$$T(p, k, b) \geqq \left\{\frac{p}{p-k}\left\{\frac{p-1}{p-k-1}\left\{\cdots\left\{\frac{b+1}{b-k+1}\right\}\cdots\right\}\right\}\right\};$$

this bound is due to Schönheim [11] and based on the inequality

$$T(p, k, b)(p-k) \geqq p \cdot T(p-1, k, b)$$

proved also by Katona, Nemetz and Simonovits [9].

Before proceeding, we mention a useful consequence of (1). Let $H$ be a $k$-graph such that $p(H)=p$, $q(H)=q$. If $q \leqq (pb^{-1})^k$ then $q < \binom{p}{k}\binom{b}{k}^{-1}$ and by (1), $\beta(H) \geqq b$. In other words, $b \leqq pq^{-1/k}$ implies $\beta(H) \geqq b$ and we have

$$\beta(H) \geqq [pq^{-1/k}]. \tag{3}$$

In fact, (3) holds for all hypergraphs $H$ such that $p(H)=p$, $q(H)=q$ and $|A| \geqq k \geqq 2$ whenever $A \in E(H)$ as an increase of the size of an edge destroys no independent set.

Let us set $R(p, k, b) = ([X]^k, \{[Y]^k : Y \in [X]^b\})$ where $|X| = p$; the $\binom{b}{k}$-graph $R(p, k, b)$ is determined uniquely up to isomorphism.

THEOREM 1.

$$T\left(\binom{p}{k}, \binom{b}{k}, \binom{p}{k} - T(p, k, b) + 1\right) \leqq \binom{p}{b}.$$

PROOF. A set $B \subset [X]^k$ is independent in $R(p, k, b)$ if and only if every $b$-tuple of elements of $X$ spans an element of $[X]^k - B$, i.e. if and only if $\beta((X, [X]^k - B)) < b$. Thus, we have

$$\beta(R(p, k, b)) = \binom{p}{k} - T(p, k, b)$$

and the desired inequality follows as $R(p, k, b)$ is a $\binom{b}{k}$-graph having $\binom{p}{k}$ vertices and $\binom{p}{b}$ edges.

By Theorem 1, every lower bound for the function $T(p, k, b)$ yields an upper bound for the same function. For instance, substituting $\binom{p}{k}$ for the first, $\binom{b}{k}$ for the second and $\binom{p}{k} - T(p, k, b) + 1$ for the third variable in the left-hand side of (1) and comparing the lower bound so obtained with the upper bound given by Theorem 1, we get an inequality

$$\binom{M}{S}\binom{N}{S}^{-1} \leqq \binom{p}{b}$$

where

$$M = \binom{p}{k}, \qquad N = \binom{p}{k} - T(p, k, b) + 1, \qquad S = \binom{b}{k}.$$

Using then the simple bound $\binom{N}{S}\binom{M}{S}^{-1} < (NM^{-1})^S$ valid for $2 \leqq S \leqq N < M$, we arrive at

COROLLARY 1A.

$$T(p, k, b) < 1 + \binom{p}{k}\left(1 - \binom{p}{b}^{-1/\binom{b}{k}}\right).$$

This is weaker than (2) for large $p$ and $b > 2k-2$ but stronger than (2) in a certain range of $p$, $k$, $b$. In particular, the next result is not deducible from (2).

COROLLARY 1B. *If $c$ is a real number such that $c \geqq 1$ then*

$$T([ck^2], k, k^2) < 1 + c^{k+1}k^2 e \ln 2.$$

This follows from Corollary 1A as

$$\binom{b}{k}\left(1 - \binom{p}{b}^{-1/\binom{b}{k}}\right) < \ln\binom{p}{b} < \ln 2^p = p \cdot \ln 2$$

and

$$\binom{p}{k}\binom{b}{k}^{-1} < \left(\frac{p-k+1}{b-k+1}\right)^k.$$

2. **Chromatic numbers—a problem of Erdös and Hajnal.** Let $H$ be a hypergraph and $r$ a positive integer such that $|A| > r$ whenever $A \in E(H)$. By an $(r, \chi)$-coloring we mean any mapping $f: V(H) \to X$ such that $|X| = \chi$ but $|f(A)| > r$ whenever $A \in E(H)$. We define $\chi(H, r)$ as the smallest $\chi$ such that there exists an $(r, \chi)$-coloring of $H$. Then $\chi(H, 1)$ is the chromatic number defined by Erdös and Hajnal [6]; the generalization into $\chi(H, r)$ has been suggested by Hell and Nešetřil.

Let $k$, $r$, $\chi$ be positive integers, $r < \min(k, \chi)$. We denote by $E(k, r, \chi)$ the least $q$ such that there exists a $k$-graph $H$ with properties $q(H) = q$, $\chi(H, r) > \chi$.

Erdös and Hajnal [5] asked for the values of $E(k, 1, 2)$; the best known bounds for $E(k, 1, 2)$ are

$$2^k(1 + 4k^{-1})^{-1} < E(k, 1, 2) < k^2 2^{k+1}.$$

The lower bound is due to Schmidt [10], the upper one to Erdös [4].

Let us realize that $p(H) \geqq [\chi r^{-1}(\beta(H)+1)]$ implies $\chi(H, r) > \chi$. Indeed, given any mapping $f: V(H) \to X$, $|X| = \chi$ there is $S \in [X]^r$ such that $|f^{-1}(S)| \geqq \{r\chi^{-1}p(H)\} \geqq \beta(H)+1$. Hence, $f^{-1}(S)$ contains an edge $A$ of $H$ and we have $|f(A)| \leqq r$, i.e. $f$ is not an $(r, \chi)$-coloring. Combining this observation with Corollary 1B we obtain

Corollary 1C. $E(k, r, \chi) \leqq \min_{b \geq k} T([\chi r^{-1} b], k, b) < 1 + (\chi r^{-1})^{k+1} k^2 e \ln 2$.

Using the technique developed by Erdös [3] in the proof of $E(k, 1, 2) > 2^{k-1}$ we obtain

Theorem 2. $E(k, r, \chi) > \binom{\chi}{r}^{-1}(\chi r^{-1})^k$.

Proof. Let $H$ be a $k$-graph such that $p(H) = p$, $q(H) = q$. Let us denote by $N$ the set of all the mappings $f: V(H) \to \{1, 2, \cdots, \chi\}$ such that there exists $A \in E(H)$ with $|f(A)| \leqq r$. Each $f \in N$ can be constructed in the following manner: Choose $A \in E(H)$ and $T \in [\{1, 2, \cdots, \chi\}]^r$, then map arbitrarily $A$ into $T$ and $V(H) - A$ into $\{1, 2, \cdots, \chi\}$. Moreover, some $f \in N$ can be obtained more than once in this way and we conclude that

$$|N| < q\binom{\chi}{r} r^k \chi^{p-k}.$$

Now, if $q \leqq \binom{\chi}{r}^{-1}(\chi r^{-1})^k$ then $|N| < \chi^p$, so there is a mapping $f: V(H) \to \{1, 2, \cdots, \chi\}$ such that $f \notin N$, i.e. $f$ is an $(r, \chi)$-coloring of $H$ and we have $\chi(H, r) \leqq \chi$.

Corollary 2A. $\lim_{k \to \infty} (E(k, r, \chi))^{1/k} = \chi r^{-1}$.

3. **Ramseyian theorems.** By a Ramseyian theorem we mean any theorem which asserts that

$$\lim_{n \to \infty} \chi(H_n, 1) = +\infty$$

for a certain sequence $H_1, H_2, \cdots$ of hypergraphs. Most of the known theorems of Ramsey's type fit into this definition. For instance, Ramsey's theorem itself can be stated as follows:

$$\lim_{p \to \infty} \chi(R(p, k, b), 1) = +\infty \qquad \text{for all } k, b.$$

When estimating the "Ramseyian numbers" associated with various Ramseyian theorems, Theorem 2 may be useful. We will illustrate this on three examples.

Let us denote by $f(k, b, \chi)$ the least integer $p$ such that $\chi(R(p, k, b), 1)>\chi$. Obviously, $R(p, k, b)$ is a $\binom{b}{k}$-graph having $\binom{p}{b}$ edges. Thus, $\binom{p}{b}<E(\binom{b}{k}, 1, \chi)$ implies $\chi(R(p, k, b), 1)\leqq\chi$ and $p<f(k, b, \chi)$. By Theorem 2, we have $\binom{p}{b}<E(\binom{b}{k}, 1, \chi)$ whenever

$$\binom{p}{b}\leqq\chi^{\binom{b}{k}-1},$$

the last inequality being satisfied for

$$p \leqq (b!)^{1/b}\chi^{(\binom{b}{k}-1)/b}$$

and we have

COROLLARY 2B (ERDÖS [2]).

$$f(k, b, \chi) > (b!)^{1/b}\chi^{(\binom{b}{k}-1)/b}.$$

If $k$, $p$ are positive integers then we denote by $W(p, k)$ the $k$-graph $H$ such that $V(H)=\{1, 2, \cdots, p\}$ and $A\in E(H)$ if, and only if, $A\in[V(H)]^k$ and the elements of $A$ form an arithmetic progression (of $k$ terms). Van der Waerden's theorem asserts that $\lim_{p\to\infty}\chi(W(p, k), 1)=+\infty$ for all $k$. Let us denote by $g(k, \chi)$ the least $p$ such that $\chi(W(p, k), 1)>\chi$. A simple computation shows that the $k$-graph $W(p, k)$ has not more than $\frac{1}{2}p^2(k-1)^{-1}$ edges. Using Theorem 2 in a similar way as above, we obtain

COROLLARY 2C (ERDÖS AND RADO [7]). $g(k, \chi)>(2(k-1)\chi^{k-1})^{1/2}$.

Finally, if $m$, $s$ are positive integers then we denote by $ER(m, s)$ the $s^2$-graph $H$ such that $V(H)=Y\times Y$ and $E(H)=\{Z\times Z:Z\in[Y]^s\}$ where $Y=\{1, 2, \cdots, m\}$. Erdös and Rado [8] have shown that $\lim_{m\to\infty}\chi(ER(m, s), 1)=+\infty$ for all $s$. Let us denote by $h(s, \chi)$ the least integer $m$ such that $\chi(ER(m, s), 1)>\chi$. Since $ER(m, s)$ has $\binom{m}{s}^2$ edges, Theorem 2 yields

COROLLARY 2D (SEE ALSO [1]). $h(s,\chi)>(s!)^{1/s}\chi^{(s^2-1)/2s}$.

Let us note that the proofs of all these estimations only depend on the number and size of the edges in the corresponding hypergraphs; at the same time, Corollary 1C shows that these bounds cannot be substantially improved unless more information about the hypergraphs in question is taken into account.

**4. A better upper bound for $E(k, 1, \chi)$.** Since our upper bound for

$E(k, r, \chi)$ was obtained as a corollary of another result, one may think about improving it by a direct method. This is what we are about to do now.

Let us set $S(p, k, \chi) = ([X]^k, E)$ where $|X| = p$ and $A \in E$ if, and only if, there exists a mapping $f: X \to \{1, 2, \cdots, \chi\}$ such that $A = \{Y \in [X]^k : |f(Y)| = 1\}$.

A set $B \subset [X]^k$ is independent in $S(p, k, \chi)$ if and only if $\chi((X, [X]^k - B), 1) > \chi$; thus, we have

$$E(k, 1, \chi) = \min_{p \geq k} \left( \binom{p}{k} - \beta(S(p, k, \chi)) \right). \tag{4}$$

$S(p, k, \chi)$ has $\binom{p}{k}$ vertices, at most $\chi^p$ edges, and each of its edges includes at least $\chi\binom{p/\chi}{k}$ vertices. By (3) and the remark following (3), we have

$$\beta(S(p, k, \chi)) \geqq \left[ \binom{p}{k} \chi^{-p/\chi \binom{p}{k}/\chi} \right]$$

or

$$\binom{p}{k} - \beta(S(p, k, \chi)) < 1 + \binom{p}{k}(1 - \chi^{-p/\chi \binom{p}{k}/\chi}) < 1 + \frac{\binom{p}{k}}{\chi\binom{p/\chi}{k}} \ln \chi^p$$

$$= 1 + \frac{\binom{p-1}{k-1}}{\binom{(p/\chi)-1}{k-1}} \ln \chi^p$$

$$< 1 + \chi^{k-1}\left(\frac{p-k+1}{p-k\chi+\chi}\right)^{k-1} p \ln \chi.$$

Setting $p = (\chi - 1)(k-1)^2 + \chi(k-1)$ and using (4) we obtain

THEOREM 3. *$E(k, 1, \chi) < 1 + \chi^k((1-1/\chi)(k-1)^2 + k - 1)e \ln \chi$.*

The case $r > 1$ would be more difficult to handle in this way. Nevertheless, since $\chi(H, 1) > \chi$ implies $\chi(H, r) > \chi r$, we have $E(k, r, \chi) < E(k, 1, \{\chi r^{-1}\})$ and Theorem 3 sometimes gives a better upper bound for $E(k, r, \chi)$ even if $r > 1$.

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