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ZERO DIVISORS AND NILPOTENT ELEMENTS IN POWER SERIES RINGS

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ABSTRACT. It is well known that a polynomial $f(X)$ over a commutative ring R with identity is nilpotent if and only if each coefficient of $f(X)$ is nilpotent; and that $f(X)$ is a zero divisor in $R[X]$ if and only if $f(X)$ is annihilated by a nonzero element of R. This paper considers the problem of determining when a power series $g(X)$ over R is either nilpotent or a zero divisor in $R[[X]]$. If R is Noetherian, then $g(X)$ is nilpotent if and only if each coefficient of $g(X)$ is nilpotent; and $g(X)$ is a zero divisor in $R[[X]]$ if and only if $g(X)$ is annihilated by a nonzero element of R. If R has positive characteristic, then $g(X)$ is nilpotent if and only if each coefficient of $g(X)$ is nilpotent and there is an upper bound on the orders of nilpotency of the coefficients of $g(X)$. Examples illustrate, however, that in general $g(X)$ need not be nilpotent if there is an upper bound on the orders of nilpotency of the coefficients of $g(X)$, and that $g(X)$ may be a zero divisor in $R[[X]]$ while $g(X)$ has a unit coefficient.

1. Introduction. It is well known that a polynomial $f(X)$ over a commutative ring R with identity is nilpotent if and only if each coefficient of $f(X)$ is nilpotent. In [**1**], McCoy establishes that a polynomial $f(X)$ is a zero divisor in $R[X]$ if and only if there is a nonzero element r of R with $rf(X)=0$. In this paper, we consider the problem of determining when a power series $g(X)$ over R is either nilpotent or a zero divisor in $R[[X]]$. We prove (Corollary 1) that if R is Noetherian, then $g(X)$ is nilpotent if and only if each coefficient of $g(X)$ is nilpotent. And if R is Noetherian, then $g(X)$ is a zero divisor of $R[[X]]$ if and only if $g(X)$ is annihilated by some nonzero element of R (Theorem 5). We establish (Theorem 1) that if R has positive characteristic, then $g(X)$ is nilpotent if and only if each coefficient of $g(X)$ is nilpotent and there is an upper bound on the orders of nilpotency of the coefficients of $g(X)$. We show by means of examples, however, that, in general, $g(X)$ need not be nilpotent if there is an upper bound on the orders of nilpotency of the coefficients

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of $g(X)$, and that $g(X)$ may be a zero divisor while $g(X)$ has a unit coefficient.

Throughout this paper, R denotes a commutative ring with identity; ω is the set of natural numbers; ω_0 is the set of nonnegative integers; Z is the set of integers; and Q is the set of rational numbers. If $f(X)=\sum_{i=0}^{\infty} f_iX^i \in R[[X]]$, we denote by A_f the ideal of R generated by the coefficients of $f(X): A_f=\{f_0, f_1, f_2, \cdots\}R$. If A is an ideal of R, we let $A[[X]]=\{f(X)=\sum_{i=0}^{\infty} f_iX^i : f_i \in A$ for each $i \in \omega_0\}$ and we define $A \cdot R[[X]]$ to be the ideal of $R[[X]]$ which is generated by A. Then $A \cdot R[[X]]=\{f(X): A_f \subseteq B$ for some finitely generated ideal B of R with $B \subseteq A\}$. It is clear that $A \cdot R[[X]] \subseteq A[[X]]$; equality holds if and only if each countably generated ideal of R contained in A is contained in a finitely generated ideal contained in A. In particular, if V is a valuation ring containing an ideal A which is countably generated but not finitely generated, then $A \cdot V[[X]] \subset A[[X]]$. Finally, we note that if A is an ideal of R, then $R[[X]]/A[[X]] \simeq (R/A)[[X]]$; hence $A[[X]]$ is a prime ideal of $R[[X]]$ if and only if A is a prime ideal of R.

2. **Nilpotent elements.** Let $f(X)=\sum_{i=0}^{\infty} f_iX^i \in R[[X]]$ and let $n \in \omega_0$; we define $f^{(n)}=\sum_{i=0}^{n} f_iX^i$. Then $f^{(n)}$ is zero or a polynomial of degree at most n.

LEMMA 1. *Let R be a commutative ring with identity having characteristic a positive prime p, and let $f(X)=\sum_{i=0}^{\infty} f_iX^i \in R[[X]]$. The following conditions are equivalent:*

(a) *$f(X)$ is nilpotent.*

(b) *There is a natural number m such that $(f_i)^m=0$ for each $i \in \omega_0$.*

(c) *There is a natural number m such that $(f^{(k)})^m=0$ for each $k \in \omega_0$.*

PROOF. (a)$\leftrightarrow$(b): This follows immediately from the fact that for each natural number n, $(f(X))^{p^n}=\sum_{i=0}^{\infty} (f_i)^{p^n} X^{ip^n}$.

(b)$\leftrightarrow$(c): This is clear since for each natural number n and for each nonnegative integer k, $(f^{(k)})^{p^n}=\sum_{i=0}^{k} (f_i)^{p^n} X^{ip^n}$.

THEOREM 1. *Let R be a commutative ring with identity having positive characteristic $n=p_1^{e_1}p_2^{e_2} \cdots p_t^{e_t}$ and let $f(X)=\sum_{i=0}^{\infty} f_iX^i \in R[[X]]$. The following conditions are equivalent:*

(a) *$f(X)$ is nilpotent.*

(b) *There is a natural number m such that $(f_i)^m=0$ for each $i \in \omega_0$.*

(c) *There is a natural number m such that $(f^{(k)})^m=0$ for each $k \in \omega_0$.*

PROOF. We let $\phi_j: R[[X]] \rightarrow R[[X]]/p_jR[[X]]$ be the natural homomorphism for $1 \leqq j \leqq t$. We note that for $1 \leqq j \leqq t$, R/p_jR has characteristic p_j.

(a)→(b): If $f(X)$ is nilpotent, then for $1 \leqq j \leqq t$, $\phi_j(f(X))$ is nilpotent in $R[[X]]/p_jR[[X]] \simeq (R/p_jR)[[X]]$. By Lemma 1, there is, for $1 \leqq j \leqq t$, a natural number m_j satisfying: For $i \in \omega_0$, $0 = (\phi_j(f_i))^{m_j} = \phi_j((f_i)^{m_j})$; that is, $(f_i)^{m_j} \in p_jR$ for each $i \in \omega_0$.

Let $m = m_1e_1 + m_2e_2 + \cdots + m_te_t$. Then for each $i \in \omega_0$,

$$(f_i)^m = (f_i^{m_1})^{e_1}(f_i^{m_2})^{e_2} \cdots (f_i^{m_t})^{e_t} \in (p_1R)^{e_1}(p_2R)^{e_2} \cdots (p_tR)^{e_t} = (0).$$

Hence (b) holds.

(b)→(a): We assume that there is a natural number m satisfying: $(f_i)^m = 0$ for each $i \in \omega_0$. Then for each j, $1 \leqq j \leqq t$, $\phi_j((f_i)^m) = (\phi_j(f_i))^m = 0$ for each $i \in \omega_0$. By Lemma 1, there is for each j, $1 \leqq j \leqq t$, a natural number m_j satisfying

$$[\phi_j(f(X))]^{m_j} = \phi_j((f(X))^{m_j}) = 0; \quad \text{that is, } (f(X))^{m_j} \in p_jR[[X]].$$

Let $m = m_1e_1 + m_2e_2 + \cdots + m_te_t$; then

$$\begin{aligned}(f(X))^m &= [(f(X))^{m_1}]^{e_1}[(f(X))^{m_2}]^{e_2} \cdots [(f(X))^{m_t}]^{e_t} \\ &\in (p_1R[[X]])^{e_1}(p_2R[[X]])^{e_2} \cdots (p_tR[[X]])^{e_t} = (0).\end{aligned}$$

Hence $f(X)$ is nilpotent.

The proof that (b)↔(c) is analogous to the proof that (a)↔(b); hence it will be omitted.

THEOREM 2. *Let R be a commutative ring with identity and let $f(X) = \sum_{i=0}^{\infty} f_iX^i \in R[[X]]$. We consider the following conditions:*

(a) *The ideal A_f is nilpotent.*

(b) *There is a natural number m which satisfies: $[A_{f^{(k)}}]^m = (0)$ for each $k \in \omega_0$.*

(c) *There is a natural number m which satisfies: $[f^{(k)}]^m = 0$ for each $k \in \omega_0$.*

(d) *There is a natural number m which satisfies: $(f_i)^m = 0$ for each $i \in \omega_0$.*

(e) *There is a natural number m which satisfies: $[f^{(k)}]^m \in (X^{k+1}) \cdot R[[X]]$ for each $k \in \omega_0$.*

(f) *$f(X)$ is nilpotent.*

We have the implications (a)↔(b)→(c)→(e)↔(f) *and* (c)→(d).

PROOF. (a)→(b): For each $k \in \omega_0$, $A_{f^{(k)}} \subseteq A_f$; hence if $(A_f)^m = (0)$, then $[A_{f^{(k)}}]^m = (0)$ for each $k \in \omega_0$.

(b)→(a): Let m be a natural number which satisfies: $[A_{f^{(k)}}]^m = (0)$ for each $k \in \omega_0$. Let $a \in (A_f)^m$; then for some $i \in \omega_0$, $a \in [A_{f^{(i)}}]^m = (0)$. Thus $(A_f)^m = (0)$.

(b)→(c): Obvious.

(c)→(d): Let m be a natural number which satisfies: $[f^{(k)}]^m=0$ for each $k\in\omega_0$. Then for each $i\in\omega_0$, $(f_i)^m=([f^{(i)}]^m)_{im}=0$.

(c)→(e): Clear.

(e)→(f): We first observe that if $i\leqq k$, then for each $m\in\omega$,

$$([f(X)]^m)_i = \sum_{\substack{e_1+e_2+\cdots+e_s=m\\ r_1e_1+r_2e_2+\cdots+r_se_s=i}} (n_{r_1r_2\cdots r_s}f_{r_1}^{e_1}f_{r_2}^{e_2}\cdots f_{r_s}^{e_s})$$

where each $n_{r_1r_2\cdots r_s}\in\omega$, implying that $([f(X)]^m)_i=([f^{(k)}]^m)_i$. For if $r_1e_1+r_2e_2+\cdots+r_se_s=i$ with each $r_i\in\omega_0$ and each $e_i\in\omega$, then $r_s\leqq i\leqq k$. Thus only the coefficients $f_0, f_1, \cdots, f_t$, where $t\leqq k$, occur in the calculation of $([f(X)]^m)_i$, whereby we obtain the above equality.

Assuming (e), let m be a natural number which satisfies: $[f^{(k)}]^m\in(X^{k+1})R[[X]]$ for each $k\in\omega_0$. Then for each $j\in\omega_0$, $([f(X)]^m)_j=([f^{(j)}]^m)_j=0$ since $[f^{(j)}]^m\in(X^{j+1})R[[X]]$. Hence $[f(X)]^m=0$ and $f(X)$ is nilpotent.

(f)→(e): We assume that $[f(X)]^m=0$; then whenever $i\leqq k$, $0=([f(X)]^m)_i=([f^{(k)}]^m)_i$. Thus $[f^{(k)}]^m\in(X^{k+1})R[[X]]$ for each $k\in\omega_0$ and (e) holds.

Corollary 1. *Let R be a commutative ring with identity and let $f(X)=\sum_{i=0}^{\infty}f_iX^i\in R[[X]]$. If A_f is a finitely generated ideal of R, then the following conditions are equivalent*:

(a) *The ideal A_f is nilpotent.*

(c) *There is a natural number m which satisfies*: $[f^{(k)}]^m=0$ *for each* $k\in\omega_0$.

(d) *There is a natural number m which satisfies*: $(f_i)^m=0$ *for each* $i\in\omega_0$.

(f) *$f(X)$ is nilpotent.*

(g) *Each coefficient of $f(X)$ is nilpotent.*

Proof. In Theorem 2, we established the implications (c)→(d) and (a)→(c)→(f). That (d)→(g) is clear. Hence it suffices to prove that (f)→(g) and that (g)→(a).

(g)→(a): If (g) holds, then each element of A_f is nilpotent. Since A_f is finitely generated, A_f is nilpotent.

(f)→(g): Let $\{P_\alpha\}$ be the collection of prime ideals of R and let N be the ideal of nilpotent elements of R; then $N=\bigcap_\alpha P_\alpha$. For each α, $P_\alpha[[X]]$ is a prime ideal of $R[[X]]$. Since $f(X)$ is nilpotent, $f(X)\in P_\alpha[[X]]$ for each α. Hence $f(X)\in\bigcap_\alpha P_\alpha[[X]]=(\bigcap_\alpha P_\alpha)[[X]]=N[[X]]$; that is, each coefficient of $f(X)$ is nilpotent.

We now give examples which show that in Theorem 2, (c)$\not\rightarrow$(a) and (d)$\not\rightarrow$(f).

EXAMPLE 1. Let $S=Z/(p)$ where p is a positive prime; let $\{X_i\}_{i\in\omega_0}$ be a countable collection of indeterminates over S; and let

$$R = S[X_0, X_1, \cdots, X_t, \cdots]/\{X_0^p, X_1^p, \cdots, X_t^p, \cdots\} \cdot S[X_0, X_1, \cdots, X_t, \cdots].$$

Let $f_i=\overline{X}_i$ and let $f(X)=\sum_{i=0}^{\infty} f_iX^i\in R[[X]]$. Then for each $k\in\omega_0$, $[f^{(k)}]^p=0$. But for each natural number n, $f_0f_1 \cdots f_{n-1}\in(A_f)^n$ and $f_0f_1 \cdots f_{n-1}\neq 0$. Thus A_f is not nilpotent. We conclude that (c)$\not\rightarrow$(a).

EXAMPLE 2. Let $n\in\omega$, $n\geqq 2$, and let

$$R = Q[X_0, X_1, \cdots, X_t, \cdots]/\{X_0^n, X_1^n, \cdots, X_t^n, \cdots\} \cdot Q[X_0, X_1, \cdots, X_t, \cdots].$$

Let $f_i=\overline{X}_i$ and let $f(X)=\sum_{i=0}^{\infty} f_iX^i\in R[[X]]$. It is clear that $(f_i)^n=0$ for each $i\in\omega_0$; hence $f(X)$ satisfies (d).

We assume that $f(X)$ is nilpotent: $[f(X)]^m=0$. Then $f_0^m=([f(X)]^m)_0=0$ so $m\geqq n$. Let k_1 be the smallest integer t for which f_0^n does not occur in every summand used in computing $([f(X)]^m)_t$. Then $0=([f(X)]^m)_{k_1}=af_0^{n-1}f_1^{m-(n-1)}$ plus other terms, each having f_0^n as a factor, where $a\in\omega$. Hence $0=([f(X)]^m)_{k_1}=af_0^{n-1}f_1^{m-(n-1)}$, implying that $m-(n-1)\geqq n$.

Let k_2 be the smallest integer t for which some summand used in computing $([f(X)]^m)_t$ has neither f_0^n nor f_1^n as a factor. Then $0=([f(X)]^m)_{k_2}=bf_0^{n-1}f_1^{n-1}f_2^{m-2(n-1)}$ plus other terms, each having either f_0^n or f_1^n as a factor, where $b\in\omega$. Hence $0=([f(X)]^m)_{k_2}=bf_0^{n-1}f_1^{n-1}\cdot f_2^{m-2(n-1)}$, implying that $m-2(n-1)\geqq n$.

We can prove inductively by this process that for each $k\in\omega$, $m-k(n-1)\geqq n$; that is, $m\geqq n+k(n-1)$. This contradicts our assumption that $m\in\omega$, showing that $f(X)$ is not nilpotent. Hence (d)$\not\rightarrow$(f).

3. Zero divisors.

LEMMA 2. *Let R be a commutative ring with identity and let $f(X)=\sum_{i=0}^{\infty} f_iX^i\in R[[X]]$. If for some natural number t, f_t is regular in R while f_i is nilpotent for $0\leqq i\leqq t-1$, then $f(X)$ is regular in $R[[X]]$.*

PROOF. We let $g(X)=\sum_{i=0}^{t-1} f_iX^i$ and $h(X)=\sum_{i=t}^{\infty} f_iX^i$; then $f(X)=g(X)+h(X)$. (We let $g(X)=0$ if $t=0$.) Since $g(X)=0$ or $g(X)$

is a polynomial of which each coefficient is nilpotent, $g(X)$ is nilpotent.

Let T denote the total quotient ring of R and let $S=T[[X]]_M$ where $M=\{X^i\}_{i=1}^{\infty}$. Then in S, we can write $h(X)=X^t h'(X)$ where $h'(X)=\sum_{i=0}^{\infty} f_{i+t}X^i$; thus $h(X)$ and $h'(X)$ are associates in S. Since $f_t=(h'(X))_0$ is regular in R, f_t is a unit of T, implying that $h'(X)$ is a unit in $T[[X]]$, hence also in S. Since $h(X)$ and $h'(X)$ are associates in S, $h(X)$ is a unit in S. Hence in S, $f(X)=g(X)+h(X)$ where $g(X)$ is nilpotent and $h(X)$ is a unit, implying that $f(X)$ is a unit, hence is regular, in S [2, Exercise 18, p. 9]. Thus $f(X)$ is regular in $R[[X]]$.

THEOREM 3. *Let R be a commutative ring with identity in which each zero divisor is nilpotent, and let $f(X)=\sum_{i=0}^{\infty} f_iX^i \in R[[X]]$. If some f_i is regular in R, then $f(X)$ is regular in $R[[X]]$.*

PROOF. This is an immediate consequence of Lemma 2, letting t be the smallest integer k for which f_k is regular in R.

COROLLARY 2. *Let R be a commutative ring with identity in which each zero divisor is nilpotent. If the ideal N of nilpotent elements of R is nilpotent, then in $R[[X]]$ each zero divisor is nilpotent.*

PROOF. Let $f(X)=\sum_{i=0}^{\infty} f_iX^i \in R[[X]]$ and assume that $f(X)$ is not nilpotent. Then A_f is not nilpotent so $A_f \not\subseteq N$; that is, not every coefficient of $f(X)$ is nilpotent. By assumption, $f(X)$ has a regular coefficient. By Theorem 3, $f(X)$ is regular in $R[[X]]$.

We observe that Corollary 2 can be restated as follows:

COROLLARY 3. *Let R be a commutative ring with identity in which* (0) *is N-primary. If N is nilpotent, then* (0) *is a primary ideal of $R[[X]]$.*

We immediately have the following:

COROLLARY 4. *Let R be a commutative ring with identity and let Q be a P-primary ideal of R. If $Q\supseteq P^k$ for some $k\in\omega$, then $Q[[X]]$ is a $P[[X]]$-primary ideal of $R[[X]]$.*

PROOF. Since $R[[X]]/Q[[X]]\simeq(R/Q)[[X]]$, it follows from Corollary 3 that $Q[[X]]$ is a primary ideal of $R[[X]]$. Also, $P^k\subseteq Q$ so that $(P[[X]])^k\subseteq P^k[[X]]\subseteq Q[[X]]$; hence $P[[X]]=\sqrt{[(P[[X]])^k]} \subseteq\sqrt{Q[[X]]}$. And clearly $\sqrt{Q[[X]]}\subseteq P[[X]]$. Hence $\sqrt{Q[[X]]} =P[[X]]$ and $Q[[X]]$ is $P[[X]]$-primary.

THEOREM 4. *Let R be a Noetherian ring with identity in which* $(0)=Q_1\cap Q_2\cap\cdots\cap Q_n$ *is a shortest primary representation, with $\sqrt{Q_i}=P_i$. Then in $R[[X]]$,* $(0)=Q_1[[X]]\cap Q_2[[X]]$

$\cap \cdots \cap Q_n[[X]]$ *is a shortest primary representation with* $\sqrt{Q_i}[[X]] = P_i[[X]]$.

PROOF. $Q_1[[X]] \cap \cdots \cap Q_n[[X]] = (Q_1 \cap \cdots \cap Q_n)[[X]] = (0)$. Further, Corollary 4 asserts that each $Q_i[[X]]$ is $P_i[[X]]$-primary. It is straightforward to verify that this primary representation of (0) in $R[[X]]$ is, in fact, irredundant.

THEOREM 5. *Let* R *be a Noetherian ring with identity in which* $(0) = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ *is a shortest primary representation of* (0) *with* $\sqrt{Q_i} = P_i$, $1 \leqq i \leqq n$. *Then for* $f(X) = \sum_{i=0}^{\infty} f_i X^i \in R[[X]]$, *these conditions are equivalent*:

(a) $f(X)$ *is a zero divisor in* $R[[X]]$.

(b) $f(X) \in P_i[[X]]$ *for some* i, $1 \leqq i \leqq n$.

(c) *There is a nonzero element* r *of* R *which satisfies* $rf(X) = 0$.

PROOF. (a)$\rightarrow$(b): This is an immediate consequence of Theorem 4 and [**3**, Corollary 3, p. 214].

(b)$\rightarrow$(c): Assuming that $f(X) \in P_i[[X]]$, this implies that $A_f \subseteq P_i$. Thus $(0):A_f \neq (0)$ by [**3**, Corollary 1, p. 214]. Let $r \in (0):A_f$, $r \neq 0$; then clearly $r \in R$ and $r \neq 0$ while $rf(X) = 0$.

(c)$\rightarrow$(a): Obvious.

We conclude with an example which shows that Theorem 5 fails when R is not Noetherian.

EXAMPLE 3.[2] Let S be a commutative ring with identity; let $\{Y, X_0, X_1, X_2, \cdots, X_t, \cdots\}$ be a set of indeterminates over S; and let

$$R = S[Y, \{X_i\}_{i=0}^{\infty}]/(X_0 Y, \{X_i - X_{i+1} Y\}_{i=0}^{\infty}).$$

Let $y = \bar{Y}$ and let $f(X) = y - X$. Then $f(X)$ has a unit coefficient, so certainly $rf(X) \neq 0$ for each nonzero element r of R. However, letting $x_i = \bar{X}_i$ and $g(X) = \sum_{i=0}^{\infty} x_i X^i$, we see that $f(X) \cdot g(X) = 0$ while $g(X) \neq 0$.

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[2] Example 3 was pointed out to the author by Professor Gilmer.