A GENERAL DIFFERENTIAL EQUATION
FOR CLASSICAL POLYNOMIALS

B. NATH

Abstract. Agrawal and Khanna [1] have derived the two partial
differential equations satisfied by the polynomial set \( B_n(x, y) \). In
this paper we shall present a generalization of these results.

Introduction. The purpose of the present paper is to derive three
partial differential equations satisfied by the polynomial set \( W_{n,\gamma,\gamma'}^{\lambda,\lambda'}(u, v, x, y) \) which is the generalization of as many as forty
classical polynomials such as Legendre polynomials, Hermite poly-
nomials, Jacobi polynomials, Gegenbauer polynomials, Sister Celine
polynomials, Bedient polynomials, generalized Bessel polynomials
etc. The polynomial set \( W_{n,\gamma,\gamma'}^{\lambda,\lambda'}(u, v, x, y) \) has been defined by
means of the generating relation

\[
(1 - mxt)^{-\lambda} \frac{F_q}{(a_p); (b_q); N\gamma t/(1 - Avt^{m'})^{\gamma'}} \times \frac{F_q}{(a'_p); (b'_q); N'y t/(1 - Avt^{m'})^{\gamma'}}
\]

\[
= \sum_{n=0}^{\infty} W_{n,\gamma,\gamma'}^{\lambda,\lambda'}(u, v, x, y)t^n,
\]

valid under the conditions given in [2]. Several other results for the
polynomial set \( W_{n,\gamma,\gamma'}^{\lambda,\lambda'}(u, v, x, y) \) have also been given in [2].

Substituting \( u^{-m} \) for \( u \) and putting \( \gamma = 0, \gamma' = 0, \lambda = 0, \lambda' = 0 \) in
(1.1), we obtain [1, p. 646 (1.1)].

Frequent use will be made of the notations given in [1].

Differential equations for \( W_{n,\gamma,\gamma'}^{\lambda,\lambda'}(u, v, x, y) \). Expanding the left
hand side of (1.1) in ascending power of \( t \), using the equality
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \psi(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n-m]} \psi(k, n-mk)
\]
and equating coefficients of \( t^n \) on both sides, we have

\[
W_n = \sum_{s=0}^{n} \sum_{k=0}^{[n-s]/m} \sum_{\rho=0}^{[s/m']} \frac{[(a_p)_k][(a'_p)_\rho]}{[(b_q)_k][(b'_q)_\rho]} \frac{(\gamma k + \lambda)_{n-s-mk}}{(s-m'p)(s-m'p)(k)(\rho)}
\]

\[
\times (\gamma' s - \gamma' m' p + \lambda') \{ m^2 \}^{n-s-mk} \{ N' y \}^{s-m'p} \{ N u \}^k \{ A v \}^\rho,
\]

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Celine polynomials, Bedient polynomials, generalized Bessel polynomials.

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where $W_n$ stands for $W_n^{\lambda,\gamma',m_m'}(u, v, x, y)$.

Let us denote

$$
\theta_1 = x \frac{\partial}{\partial x}, \quad \theta_2 = y \frac{\partial}{\partial y}, \quad \theta_3 = u \frac{\partial}{\partial u} \quad \text{and} \quad \theta_4 = v \frac{\partial}{\partial v}.
$$

Let us consider

$$
\theta_2 \{ \theta_2 + (b'_2') - 1 \} \{ \gamma' \theta_2 + \lambda' - \gamma' \gamma' \} \{ \gamma \theta_3 + \theta_4 + \lambda \} W_n.
$$

We have

$$
\theta_2 \{ \theta_2 + (b'_2') - 1 \} \{ \gamma' \theta_2 + \lambda' - \gamma' \gamma' \} \{ \gamma \theta_3 + \theta_4 + \lambda \} W_n
$$

$$
= \sum_{s=0}^{n-1} \sum_{k=0}^{[n-s/m]} \sum_{\rho=0}^{[s/m']} \left\{ \frac{\lambda + \gamma k + n - s - m k}{(n-s-m k)(s-m' \rho)(\rho)[(b'_2')]_{s-m' \rho}} \times \{ \lambda + \gamma k + n - s - m k \} \{ (a_p)_{s-m' \rho} (\gamma' s - \gamma' m' \rho + \lambda') \} \{ mx \}^{n-s-m k} \{Ny\}^{s-m' \rho} \{ Nu \}^k \{ Av \}^p \right\}
$$

$$
= \sum_{s=0}^{n-1} \sum_{k=0}^{[n-s-1/m]} \sum_{\rho=0}^{[s+1/m']} \left\{ \frac{(a'_p) + s - m' \rho}{(n-s-m k)(s-m' \rho)(\rho)[(b'_2')]_{s-m' \rho}} \times \{ (a_p)_{s-m' \rho} (\gamma' s - \gamma' m' \rho + \lambda') \} \{ mx \}^{n-s-m k} \{Ny\}^{s-m' \rho} \{ Nu \}^k \{ Av \}^p \right\}
$$

Therefore,

$$
\begin{align*}
&\left[ mx \left\{ \theta_2 \prod_{i=1}^{q'} (\theta_2 + b'_i - 1)(\gamma' \theta_2 + \lambda' - \gamma') \gamma' \theta_3 + \theta_4 + \lambda \right\} \\
&\quad - N' \gamma \theta_1 \prod_{i=1}^{p'} (\theta_2 + a'_i)(\gamma' \theta_2 + \lambda' + \theta_4) \right] W_n \\
&= 0,
\end{align*}
$$

which is one of the differential equations for the polynomial set $W_n^{\lambda,\gamma',m_m'}(u, v, x, y)$.

Similarly, it can be also shown that the other partial differential equations for $W_n^{\lambda,\gamma',m_m'}(u, v, x, y)$ are given by
\[
(2.3) \quad \left[ (N'y)^m \prod_{i=1}^{p'} (1 - a_i - \theta_2 - m')_m '(1 - \lambda' - \gamma' \theta_2 - \gamma'm')_m' \\
\times (\theta_4 - \lambda' - \gamma' \theta_2 - \gamma'm')_m' - (-1)^{p'm'} \\
\times (A v) (\gamma \theta_2 + \lambda' + \theta_4) (1 + \theta_2 - m')_m' \prod_{i=1}^{q'} (b_i' + \theta_2 - m')_m' \right] W_n \\
= 0.
\]

and

\[
(2.4) \quad \left[ (mx)^m \prod_{i=1}^{q} (\theta_3 + b_i - 1)(\lambda + \gamma \theta_3 - \gamma),(\lambda + \theta_1 + \gamma \theta_3 - \gamma)_m \\
- Nu(1 + \theta_1 - m)_m'(\lambda + \theta_1 + \gamma \theta_3)_m' \prod_{i=1}^{p} (\theta_3 + a_i) \right] W_n \\
= 0.
\]

The equations (2.2), (2.3) and (2.4) are the partial differential equations satisfied by the polynomial set \( W_{n, \gamma}^{\lambda, \lambda'}^{m,m'}(u,v,x,y) \).

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References


Banaras Hindu University, Varanasi 5, India