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A GENERAL DIFFERENTIAL EQUATION FOR CLASSICAL POLYNOMIALS

B. NATH

ABSTRACT. Agrawal and Khanna [1] have derived the two partial differential equations satisfied by the polynomial set $B_n(x, y)$. In this paper we shall present a generalization of these results.

Introduction. The purpose of the present paper is to derive three partial differential equations satisfied by the polynomial set $W_{n;\gamma,\gamma'}^{\lambda,\lambda';m,m'}(u, v, x, y)$ which is the generalization of as many as forty classical polynomials such as Legendre polynomials, Hermite polynomials, Jacobi polynomials, Gegenbauer polynomials, Sister Celine polynomials, Bedient polynomials, generalized Bessel polynomials etc. The polynomial set $W_{n;\gamma,\gamma'}^{\lambda,\lambda';m,m'}(u, v, x, y)$ has been defined by means of the generating relation

$$
\begin{aligned}
&(1 - mxt)^{-\lambda}{}_pF_q[(a_p); (b_q); Nut^m/(1 - mxt)^{\gamma}](1 - Avt^{m'})^{-\lambda'} \\
&\qquad \times {}_{p'}F_{q'}[(a'_{p'}); (b'_{q'}); N'yt/(1 - Avt^{m'})^{\gamma'}] \\
&\qquad = \sum_{n=0}^{\infty} W_{n;\gamma,\gamma'}^{\lambda,\lambda';m,m'}(u, v, x, y)t^n,
\end{aligned} \tag{1.1}
$$

valid under the conditions given in [2]. Several other results for the polynomial set $W_{n;\gamma,\gamma}^{\lambda,\lambda';m,m'}(u, v, x, y)$ have also been given in [2].

Substituting u^{-m} for u and putting $\gamma=0$, $\gamma'=0$, $\lambda=0$, $\lambda'=0$ in (1.1), we obtain [1, p. 646 (1.1)].

Frequent use will be made of the notations given in [1].

Differential equations for $W_{n;\lambda,\lambda'}^{\lambda,\lambda';m,m'}(u, v, x, y)$. Expanding the left hand side of (1.1) in ascending power of t, using the equality $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\psi(k, n) = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\psi(k, n-mk)$ and equating coefficients of t^n on both sides, we have

$$
\begin{aligned}
W_n = \sum_{s=0}^{n}\sum_{k=0}^{[n-s/m]}\sum_{\rho=0}^{[s/m']} &\frac{[(a_p)]_k[(a'_{p'})]_{s-m'\rho}(\gamma k+\lambda)_{n-s-mk}}{[(b_q)]_k[(b'_{q'})]_{s-m'\rho}(n-s-mk)(s-m'\rho)(k)(\rho)} \\
&\times(\gamma's-\gamma'm'\rho+\lambda')_\rho\{mx\}^{n-s-mk}\{N'y\}^{s-m'\rho}\{Nu\}^k\{Av\}^\rho,
\end{aligned} \tag{2.1}
$$

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where W_n stands for $W_{n;\gamma,\gamma'}^{\lambda,\lambda';m,m'}(u, v, x, y)$.

Let us denote

$$\theta_1 = x\frac{\partial}{\partial x}, \qquad \theta_2 = y\frac{\partial}{\partial y}, \qquad \theta_3 = u\frac{\partial}{\partial u} \quad \text{and} \quad \theta_4 = v\frac{\partial}{\partial v}\cdot$$

Let us consider

$$\theta_2\{\theta_2 + (b_{q'}') - 1\}(\gamma'\theta_2 + \lambda' - \gamma')_{\gamma'}\{\gamma\theta_3 + \theta_1 + \lambda\}W_n.$$

We have

$$\begin{aligned}
&\theta_2\{\theta_2+(b_{q'}')-1\}(\gamma'\theta_2+\lambda'-\gamma')_{\gamma'}\{\gamma\theta_3+\theta_1+\lambda\}W_n \\
&= \sum_{s=1}^{n}\sum_{k=0}^{[n-s/m]}\sum_{\rho=0}^{[s/m']} \\
&\cdot\frac{\{s-m'\rho\}\{s-m'\rho+(b_{q'}')\}(\gamma'\{s-m'\rho\}+\lambda'-\gamma')_{\gamma'}\{\lambda+\gamma k+n-s-mk\}}{(n-s-mk)(s-m'\rho)(k)(\rho)[(b_q)]_k[(b_{q'}')]_{s-m'\rho}} \\
&\times\{\lambda+\gamma k+n-s-mk\}[(a_p)]_k[(a_{p'}')]_{s-m'\rho}(\gamma k+\lambda)_{n-s-mk}(\gamma's-\gamma'm'\rho+\lambda')_\rho \\
&\times\{mx\}^{n-s-mk}\{Ny\}^{s-m'\rho}\{Nu\}^k\{Av\}^\rho \\
&= \sum_{s=0}^{n-1}\sum_{k=0}^{[n-s-1/m]}\sum_{\rho=0}^{[s+1/m']} \\
&\cdot\frac{\{(a_{p'}')+s-m'\rho\}(\gamma'\{s-m'\rho\}+\lambda')_{\rho+\gamma'}\{n-s-mk\}(\gamma k+\lambda)_{n-s-mk}}{(n-s-mk)(s-m'\rho)(\rho)(k)[(b_q)]_k[(b_{q'}')]_{s-m'\rho}} \\
&\times[(a_p)]_k[(a_{p'}')]_{s-m'\rho}\{N'y/mx\}\{mx\}^{n-s-mk}\{Ny\}^{s-m'\rho}\{Nu\}^k\{Av\}^\rho \\
&= \{N'y/mx\}\theta_1\{\theta_2+(a_{p'}')\}(\gamma'\theta_2+\lambda'+\theta_4)_{\gamma'}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left[mx\left\{\theta_2\prod_{i=1}^{q'}(\theta_2 + b_i' - 1)(\gamma'\theta_2 + \lambda' - \gamma')_{\gamma'}(\gamma\theta_3 + \theta_1 + \lambda)\right\}\right. \\
&\qquad\qquad \left. - N'y\theta_1\prod_{i=1}^{p'}(\theta_2 + a_i')(\gamma'\theta_2 + \lambda' + \theta_4)\right]W_n \\
&\qquad\qquad = 0,
\end{aligned} \tag{2.2}$$

which is one of the differential equations for the polynomial set $W_{n;\gamma,\gamma'}^{\lambda,\lambda';m,m'}(u, v, x, y)$.

Similarly, it can be also shown that the other partial differential equations for $W_{n;\gamma,\gamma'}^{\lambda,\lambda';m,m'}(u, v, x, y)$ are given by

$$
\begin{aligned}
&\Big[(N'y)^{m}\theta_4 \prod_{i=1}^{p'} (1 - a'_i - \theta_2 - m')_{m'}(1 - \lambda' - \gamma'\theta_2 - \gamma' m')_{\gamma' m'} \\
(2.3)\qquad &\times (\theta_4 - \lambda' - \gamma'\theta_2 - \gamma' m')_{\gamma' m'} - (-1)^{p'm'} \\
&\times (Av)(\gamma\theta_2 + \lambda' + \theta_4)(1 + \theta_2 - m')_{m'} \prod_{i=1}^{q'} (b'_i + \theta_2 - m')_{m'}\Big] W_n \\
&= 0.
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big[(mx)^{m}\theta_3 \prod_{i=1}^{q} (\theta_3 + b_i - 1)(\lambda + \gamma\theta_3 - \gamma)_{\gamma}(\lambda + \theta_1 + \gamma\theta_3 - \gamma)_m \\
(2.4)\qquad &\quad - Nu(1 + \theta_1 - m)_m(\lambda + \theta_1 + \gamma\theta_3)_{\gamma'} \prod_{i=1}^{p} (\theta_3 + a_i)\Big] W_n \\
&\quad = 0.
\end{aligned}
$$

The equations (2.2), (2.3) and (2.4) are the partial differential equations satisfied by the polynomial set $W_{n;\gamma,\gamma'}^{\lambda,\lambda';m,m'}(u, v, x, y)$.

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BANARAS HINDU UNIVERSITY, VARANASI 5, INDIA