PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 3, March 1971

APPROXIMATION BY HOMEOMORPHISMS AND SOLUTION OF P. BLASS PROBLEM ON PSEUDO-ISOTOPY

W. HOLSZTYŃSKI

ABSTRACT. For every map of $f:S^1\to S^1=\{z\in C:|z|=1\}$ of degree 1, existence of a pseudo-isotopy $h:S^1\times I\to R=\{z\in C:|z|\geqq 1\}$ such that $h(z,0)=z$ and $h(z,1)=f(z)$ is established. On the other hand (i) maps of I^n into $I^n\times 0\subset E^{n+1}$ cannot be, in general, uniformly approximated by homeomorphic embeddings of I^n in E^{n+1} for $n>1$, and (ii) maps of S^n into $S^n\subset E^n$ of degree 1 cannot be, in general, extended to a pseudo-isotopy of S^n into E^{n+1}.

P. Blass asked me: Can every mapping $g:S^n\to S^n$ of degree 1 be obtained by a pseudo-isotopy in Euclidean space E^{n+1} from an embedding? Does there hold an analogous assertion for mappings of S^n into itself of other degree?

We will show that the answer is positive for $n=1$ (see §1) and negative for $n>1$ (see §4).

1. 1-dimensional case of a map of degree 1. First we will describe the most intuitive case. Some more general and stronger results are contained in §2.

Let C be the complex plane,

$$S^1=\{z\in C:|z|=1\},$$
$$R=\{z\in C:|z|\geqq 1\}.$$

(1.1) THEOREM. *Let $f:S^1\to S^1$ be a map of degree 1. Then there exists a homotopy $F:S^1\times I\to R$ such that $F|S^1\times\{t\}$ is a homeomorphic embedding for every $t\in I\cdot\{1\}$, and $F(z,1)=f(z)$ for every $z\in S^1$.*

PROOF. Instead of pair (R, S^1) we will consider the homeomorphic pair $(S^1\times E^+, S^1\times\{0\})$, where E^+ is the set of all nonnegative reals. Let $h:S^1\times I\to S^1$ be a homotopy such that $h(z,0)=2$ and $h(2,1)=f(z)$ for $z\in S^1$. Next, let

$$u(z,t)=h(z,t)z^{-1}\quad\text{for}\quad(z,t)\in S^1\times I. \tag{1.2}$$

Then $u(S^1\times\{0\})=1$, so that there exists a mapping $v:S^1\times I\to E^+$

Received by the editors January 7, 1970.

AMS 1970 *subject classifications*. Primary 57A35, 57A05; Secondary 55C25.

Key words and phrases. Pseudo-isotopy, generalizated pseudo-isotopy, isotopy domination, approximation by homeomorphisms, maps of degree 1, sphere S^1 and S^n, complex plane C, Euclidean space E^{n+1}.

such that $u=e^{2\pi i\cdot v}$. Homotopy $F_0\colon S^1\times I\to S^1\times E^+$ given by formula $F_0=h\Delta v$ i.e. $F_0(z, t)=(h(z, t), v(z, t))$ has the following two properties: if $t<1$ and $z_1\neq z_2$ and $h(z_1, t)=h(z_2, t)$ then $u(z_1, t)\neq u(z_2, t)$ and $F_0(z_1, t)\neq F_0(z_2, t)$. Thus $F_0|S^1\times\{t\}$ is a homeomorphic embedding for $t<1$. The second property is:

$$F_0(z, 1) = (f(z), v(z, 1)).$$

Thus the required homotopy $F\colon S^1\times I\to S^1\times E^+$ can be given by

$$\begin{aligned} F(z, t) &= F_0(z, t/2) && \text{for } 0 \leqq t \leqq \tfrac{1}{2}, \\ &= (f(z), (2-2t)\cdot v(z, 1)) && \text{for } \tfrac{1}{2} \leqq t \leqq 1. \end{aligned}$$

REMARK. A homotopy F on X such that $F|X\times\{t\}$ is a homeomorphic embedding for $0\leqq t<1$ is said to be a pseudo-isotopy (compare F from Theorem (1.1)).

2. 1-dimensional case of a map of degree n.

We will give a generalization of pseudo-isotopy.

(2.1) DEFINITION. Given topological spaces X, Y, a mapping $g\colon X\to Y$ is isotopically dominated by a mapping $f\colon X\to Y$ iff there exists a homotopy $F\colon X\times I\to Y$ such that

(i) $F(x, 0)=f(x)$ and $F(x, 1)=g(x)$.

(ii) If $0\leqq t<1$ and $f(p)\neq f(q)$ then $F(p, t)\neq F(q, t)$ for every p, $q\in X$.

(iii) If $\frac{1}{2}\leqq t<1$ then $F(p, t)=F(q, t)$ iff $F(p, \frac{1}{2})=F(q, \frac{1}{2})$, for every p, $q\in X$.

The homotopy F will be called a pseudo-isotopy. In the case of a homeomorphic embedding f and compact X the homotopy F is a pseudo-isotopy in the usual sense.

Let us remark that in such a case also the homeomorphic embedding f is isotopically dominated by g (we can define a respective pseudo-isotopy G by $G(x, t)=F(x, 1-t)$).

(2.2) THEOREM. *If, for f, $g\colon S^1\to S^1$, R ord $f=$ ord g then g is isotopically dominated by f in R.*

PROOF. Let $h\colon S^1\times I\to S^1$ be a homotopy that connects f and g, i.e. $h(z, 0)=f(z)$ and $h(z, 1)=g(z)$. Next, let

$$u(z, t) = h(z, t)(f(z))^{-1} \quad \text{for} \quad (z, t) \in S^1 \times I \quad \text{(compare (1.2))}.$$

Then $u(S^1\times\{0\})=1$ so that there exists a mapping $v\colon S^1\times I\to E^1$ into the real line E^1 such that $u=e^{2\pi i\cdot v}$. Then the desired pseudo-isotopy $F\colon S^1\times I\to R$ is given by

$$(2.3)\quad \begin{aligned} F(z, t) &= v'(z, 2t)\cdot h(z, 2t) && \text{for } 0 < t < \tfrac{1}{2}, \\ &= (2(1-t)v'(z, t) + 2t - 1)\cdot g(z) && \text{for } \tfrac{1}{2} \leqq t \leqq 1, \end{aligned}$$

where $v'(z, t) = 1 + v(z, t) - \inf_{x\in S^1} v(x, t)$.

Indeed, F is a well-defined mapping and condition (i) holds. Next, if $f(p) \neq f(q)$ and $h(p, t) = h(q, t)$ then $u(p, t) \neq u(q, t)$, and consequently $v'(p, t) \neq v'(p, t)$. But if $h(p, t') \neq h(q, t')$ or $v'(p, t') \neq v'(q, t')$ for $t' = \min(2t, 1)$ and $t < 1$, then $F(p, t) \neq F(q, t)$. Thus condition (ii) holds. It is easy to see that condition (iii) also holds.

(2.4) COROLLARY. *If $g: S^1 \to S^1$ is a mapping of order* 1 *then there exists a homotopy $F: S^1 \times I \to R$ such that*

(*) *$F(z, 0) = z$ and $F(z, 1) = g(z)$ for every $z \in S^1$,*

(**) *$F| S^1 \times \{t\}$ is a homeomorphic embedding for $0 \leqq t < 1$.*

Looking for $F| S^1 \times [\frac{1}{2}; 1]$ at (2.3) it is easy to obtain the following

(2.5) COROLLARY. *Let* ord $f =$ ord g *for $f, g: S1 \to S1$. Then there exist $f_1: S^1 \to S^1 \times I$ and $f_2: f_1(S^1) \to S^1$ such that $f = f_2 \circ f_1$ and $g = p \cdot f_1$, where $p = S^1 \times I \to S^1$ is the projection ($p(z, t) = z$).*

(2.6) COROLLARY. *If* ord $g = 1$ *for $g: S^1 \to S^1$, then there exists a homeomorphic embedding $f_1 = S^1 \to S^1 \times I$ such that $g = p \circ f_1$.*

3. **Approximations by homeomorphisms.** Let $Q^n = \{x \in E^n : |x| \leqq 1\}$, $S^{n-1} = \dot{Q}^n$ and let $\varphi: S^{n-1} \to S^{n-1} \times \{0\} \subset E^{n+1}$ be a continuous mapping. Next let X_1 be a space obtained from Q^n by identification of points x, x' such that $\varphi(x) = \varphi(x')$ and let X_2 be a space obtained from $S^{n-1} \times I$ by identification of points $(x, 0)$ and $(\varphi(x), 1)$. Then X_1, X_2 are the compact metrizable spaces such that

$$H_{n-1}(X_1) = Z_k \quad \text{and} \quad H_{n-1}(X_2) = Z_{k-1}$$

where $k =$ ord φ (we shall consider Čech homology theory).

(3.1) THEOREM. *Under the assumption* $|$ord $\varphi| > 1$, *there does not exist a sequence of homeomorphic embeddings of Q^n into E^{n+1} which is uniformly convergent to a mapping $g: Q^n \to E^{n+1}$, such that $g(x) = (\varphi(x), 0)$ for $x \in S^{n-1}$ and $g^{-1}(S^{n-1} \times \{0\}) = S^{n-1}$.*

PROOF. Let $f: Q^n \to E^{n+1}$ be a homeomorphic embedding of Q^n into E^{n+1} such that

$$\epsilon = \epsilon(f) = \max_{x \in Q^{n-1}} |f(x) - g(x)| < 1.[1]$$

[1] In fact, we think that there does not exist such f.

Then we define $h_f: X_1 \to E^{n+1}$ as follows

$$h_f(x) = f\left(\frac{x}{1-\epsilon}\right) \qquad \text{for } |x| \leqq 1 - \epsilon$$
$$= \frac{1-|x|}{\epsilon} \cdot f\left(\frac{x}{|x|}\right) + \left(1 - \frac{1-|x|}{\epsilon}\right) \cdot g(x).$$

Now, if for a sequence $f_1, f_2, \cdots, \epsilon = \epsilon(f_n) \to 0$ then the mappings $h_f: X_1 \to R^{n+1}$ are arbitrarily fine (i.e. under a metric in X_1 the mappings h_{f_n} are δ_n-mappings with $\delta_n \to 0$). For this reason $H_{n-1}(h_f(X_1))$ contains a cyclic element of order $k = \text{ord } \varphi$, for an embedding f (to prove it see for instance [1, p. 39] and [2]). But $h_f(X_1)$ is a subspace of E^{n+1}. This contradiction shows the truth of the theorem.

(3.2) THEOREM. *Let $g: S^{n-1} \times I \to E^{n+1}$ be a mapping such that $g(x, 0) = (x, 0)$, $g(x, 1) = (\varphi(x), 0)$ for every $x \in S^{n-1}$, and $g^{-1}(S^{n-1} \times \{0\}) = S^{n-1} \times \{0, 1\}$. Then, under the assumption $|\text{ord } \varphi - 1| > 1$, there does not exist a sequence of homeomorphic embeddings of $S^{n-1} \times I$ into E^{n+1} which is uniformly convergent to g.*

PROOF. Let $f: S^{n-1} \times I \to E^{n+1}$ be a homeomorphic embedding such that

$$\epsilon = \epsilon(f) = \max_{(x,t) \in S^{n-1} \times I} |f(x, t) - g(x, t)| < \tfrac{1}{2}.$$

Then we define $h_f: X_2 \to E^{n+1}$ as follows:

$$h_f(x, t) = f(x, t) \qquad \text{for } \epsilon \leqq t \leqq 1 - \epsilon,$$
$$= \frac{t}{\epsilon} f(x, t) + \left(1 - \frac{t}{\epsilon}\right) g(x, t) \qquad \text{for } 0 \leqq t \leqq \epsilon,$$
$$= \frac{1-t}{\epsilon} f(x, t) + \frac{t - 1 + \epsilon}{\epsilon} g(x, t) \qquad \text{for } 1 - \epsilon \leqq t \leqq 1.$$

Now we can repeat the arguments from the proof of Theorem (3.1).

4. n-dimensional case, $n \geqq 2$. Let S^n be the unit sphere of Euclidean space $E^{n+1} = E^n \times E^1$, and let $g: S^n \to S^n$ be given by

$$g(x, t) = (s \cdot x, 2t + 1) \qquad \text{for } -1 \leqq t \leqq 0 \quad \text{and} \quad s = \frac{1 - (2t+1)^2}{|x|},$$
$$= (s \cdot \varphi(x), 1\text{–}2 \min(t, 1 - t))$$
$$\text{for } 0 \leqq t \leqq 1 \quad \text{and} \quad s = 1 - 4(t - \tfrac{1}{2})^2$$

where $\varphi: S^{n-1} \to S^{n-1}$ is a mapping of order $\neq 0, 1$. It is easy to see that ord $g=1$ i.e. that g is homotopic to the identity mapping. But there does not exist a pseudo-isotopy for g i.e. such a homotopy $F: S^n \times I \to E^{n+1}$ that $F|S^n \times \{t\}$ is a homeomorphism for $0 \leqq t < 1$ and that $F(x, 1) = g(x)$ for every $x \in S^n$. Furthermore, let

$$P_1 = \{(x, t) \in S^n : t \geqq \tfrac{1}{2}\}, \qquad P_2 = \{(x, t) \in S^n : |t| \leqq \tfrac{1}{2}\}$$

and $g_i = g|P_i$ for $i = 1, 2$. We denote also by $p: E^{n+1} \to E^n$ the projection given by $p(x, t) = x$. Then the following lemmas hold; these are the consequences of the result of §3.

(4.1) Lemma. *If* $|\text{ord } \varphi| > 1$ *then there does not exist a sequence of homeomorphic embeddings* $f_n: P_1 \to E^{n+1}$ *which is uniformly convergent to* $g_1|P_1$.

(4.2) Lemma. *If* $|\text{ord } \varphi - 1| > 1$ *then there does not exist a sequence of homeomorphic embeddings* $f_n: P_2 \to E^{n+1}$ *which is uniformly convergent to* $g_2|P_2$.

(4.3) Corollary. *For every* $n \geqq 2$ *there exists a mapping* $g: S^n \to S^n \subseteq E^{n+1}$ *of order* 1, *that is not isotopically dominated by a homeomorphic embedding.*

References

1. K. Borsuk, *Theory of retracts*, Monografie Mat., Tom 44, PWN, Warsaw, 1967. MR **35** #7306.

2. W. Holsztyński and S. Iliadis, *Approximation of multi-valued by single-valued mappings and some applications*, Bull. Polon. Acad. Sci. Sér. Sci. Math. Astronom. Phys. **16** (1968), 765–769. MR **39** #929.

University of Michigan, Ann Arbor, Michigan 48104.