SUSPENDING HOMOTOPY 3 SPHERES AND EMBEDDING MAPPING CYLINDERS IN S^4

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Abstract. A property of maps between closed 3-manifolds, implied by cellularity and implying UV°, is that the mapping cylinder embed locally in S^4. It is not clear what topological properties are preserved under such maps. In the present note, we show that a closed 3-manifold admits such a map onto S^3 if and only if its suspension is S^3.

In [6], it was shown that if f is a map of the closed 3-manifold M onto itself such that the mapping cylinder Z_f of f embeds in S^4, then Z_f ∼ M×[0, 1]. The situation with regard to maps between (possibly) different manifolds was unclear, and the following question was asked: If f is a mapping of the closed 3-manifold M onto S^3 such that Z_f embeds in S^4, must M be homeomorphic to S^3?

Notation. If f: X→Y is a map, Z_f denotes its mapping cylinder (with X and Y identified as subsets of Z_f, as usual). Σ(X) denotes the suspension of X, and "≈" means topological equivalence. I denotes the interval [0, 1].

Theorem 1. Let M and N be closed 3-manifolds, f: M→N an onto map. If Z_f embeds (locally) in S^4, then Σ(M) ≈ Σ(N).

Proof. Let V be an open set in N, U=f^-1(V), such that Z_f |V embeds in S^4. Using the arguments in [6], we see that V is locally collared in Z_f |V.

(Sketch of proof. A result of Wilder [8] shows that Z_f |V is lc^∞ mod V, so f is uv^∞-trivial; applying Wright [9], we obtain a locally finite subset F of V such that each f^-1(y) is cellular in U, y∈V−F; then, by a result of Armentrout [1], it follows that V is locally collared in Z_f |V at each point of V−F; finally, a result of Kirby [4] shows that V is locally collared in Z_f |V at each point of F.) Thus, N is locally collared in Z_f.

An engulfing-monotone union argument now shows that Z_f−M ≈ N×(0, 1] and hence that M×(0, 1) ≈ Z_f−M−N ≈ N×(0, 1) (see [5, Theorem 1.1]). For compact X, Σ(X) is homeomorphic to the 2-point compactification of X×(0, 1), so the proof is complete.

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Theorem 2. If $M$ is a closed 3-manifold such that $\Sigma(M) \approx S^4$, then there exists a map $f$ of $M$ onto $S^3$ such that $Z_f$ embeds in $S^4$.

Proof. Let $M_0$ be the complement of an open 3-simplex of $M$, and let $f$ be a map of $M$ onto $S^3$ whose only nondegenerate point-inverse is $M_0$. We will show that $Z_f$ embeds in the cone $M \ast v$.

Let $g: M \times I \to M \times I$ be an embedding such that $g(x, 0) = (x, 0)$ and $g(x \times I) \subseteq (x \times I)$ for all $x$ in $M$, and such that $g(x \times I) = x \times I$ if and only if $x \in M_0$. Let $q: M \times I \to Z_f$ be the quotient map (whose only nondegenerate point-inverse is $M_0 \times 1$) and let $p: M \times I \to M \ast v$ be the map which shrinks $M \times 1$ to the vertex $v$. Define $h$ by

$$h = pgq^{-1}: Z_f \to M \ast v.$$ 

One can easily check that $h$ is a well-defined, continuous, one-one function.

Corollary. The following are equivalent statements about a closed 3-manifold $M$.

(a) There exists a map $f$ of $M$ onto $S^3$ such that $Z_f$ embeds in $S^4$;
(b) $\Sigma(M) \approx S^4$.

Remark 1. If $M^4$ is a closed manifold homotopy equivalent to $S^4$, it is known that $\Sigma(M^4) \approx S^5$ (see Hirsch [3] and Harley [2]). Moreover, Siebenmann [7] has recently shown that $\Sigma^k(M^3) \approx S^5$ for any homotopy 3-sphere $M^3$. Thus, if $M^m$ is any (closed manifold) homotopy $m$-sphere, $k > 0$, and $k + m \not\equiv 4$, then $\Sigma^k(M^m) \approx S^{m+k}$. The case $k = 1$, $m = 3$, is lacking a solution.

Remark 2. The above proofs can easily be modified to yield the following results. If $f: M \to N$ is an onto map between closed 3-manifolds such that $Z_f$ embeds locally in $S^4$ then $M$ is homeomorphic to the connected sum $N \# H$, where $H$ is a closed 3-manifold and $\Sigma(H) \approx S^4$. Conversely, if $M = N \# H$ where $N$ and $H$ are closed 3-manifolds and $\Sigma(H) \approx S^4$, then there exists a map of $M$ onto $N$ whose mapping cylinder is a topological manifold with boundary $M \cup N$.

References


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