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SUSPENDING HOMOTOPY 3-SPHERES AND EMBEDDING MAPPING CYLINDERS IN S^4

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ABSTRACT. A property of maps between closed 3-manifolds, implied by cellularity and implying UV^∞, is that the mapping cylinder embed locally in S^4. It is not clear what topological properties are preserved under such maps. In the present note, we show that a closed 3-manifold admits such a map onto S^3 if and only if its suspension is S^4.

In [6], it was shown that if f is a map of the closed 3-manifold M onto itself such that the mapping cylinder Z_f of f embeds in S^4, then $Z_f \approx M \times [0, 1]$. The situation with regard to maps between (possibly) different manifolds was unclear, and the following question was asked: If f is a mapping of the closed 3-manifold M onto S^3 such that Z_f embeds in S^4, must M be homeomorphic to S^3?

Notation. If $f: X \to Y$ is a map, Z_f denotes its mapping cylinder (with X and Y identified as subsets of Z_f, as usual). $\Sigma(X)$ denotes the suspension of X, and "$\approx$" means topological equivalence. I denotes the interval $[0, 1]$.

THEOREM 1. *Let M and N be closed 3-manifolds, $f: M \to N$ an onto map. If Z_f embeds (locally) in S^4, then $\Sigma(M) \approx \Sigma(N)$.*

PROOF. Let V be an open set in N, $U = f^{-1}(V)$, such that $Z_{f|U}$ embeds in S^4. Using the arguments in [6], we see that V is locally collared in $Z_{f|U}$.

(*Sketch of proof.* A result of Wilder [8] shows that $Z_{f|U}$ is lc^∞ mod V, so f is uv^∞-trivial; applying Wright [9], we obtain a locally finite subset F of V such that each $f^{-1}(y)$ is cellular in U, $y \in V - F$; then, by a result of Armentrout [1], it follows that V is locally collared in $Z_{f|U}$ at each point of $V - F$; finally, a result of Kirby [4] shows that V is locally collared in $Z_{f|U}$ at each point of F.) Thus, N is locally collared in Z_f.

An engulfing-monotone union argument now shows that $Z_f - M \approx N \times (0, 1]$ and hence that $M \times (0, 1) \approx Z_f - M - N \approx N \times (0, 1)$ (see [5, Theorem 1.1]). For compact X, $\Sigma(X)$ is homeomorphic to the 2-point compactification of $X \times (0, 1)$, so the proof is complete.

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THEOREM 2. *If M is a closed 3-manifold such that $\Sigma(M) \approx S^4$, then there exists a map f of M onto S^3 such that Z_f embeds in S^4.*

PROOF. Let M_0 be the complement of an open 3-simplex of M, and let f be a map of M onto S^3 whose only nondegenerate point-inverse is M_0. We will show that Z_f embeds in the cone $M * v$.

Let $g: M \times I \rightarrow M \times I$ be an embedding such that $g(x, 0) = (x, 0)$ and $g(x \times I) \subset (x \times I)$ for all x in M, and such that $g(x \times I) = x \times I$ if and only if $x \in M_0$. Let $q: M \times I \rightarrow Z_f$ be the quotient map (whose only nondegenerate point-inverse is $M_0 \times 1$) and let $p: M \times I \rightarrow M * v$ be the map which shrinks $M \times 1$ to the vertex v. Define h by

$$h = pgq^{-1}: Z_f \rightarrow M * v.$$

One can easily check that h is a well-defined, continuous, one-one function.

COROLLARY. *The following are equivalent statements about a closed 3-manifold M.*

(a) *There exists a map f of M onto S^3 such that Z_f embeds in S^4*;

(b) $\Sigma(M) \approx S^4$.

REMARK 1. If M^4 is a closed manifold homotopy equivalent to S^4, it is known that $\Sigma(M^4) \approx S^5$ (see Hirsch [3] and Harley [2]). Moreover, Siebenmann [7] has recently shown that $\Sigma^2(M^3) \approx S^5$ for any homotopy 3-sphere M^3. Thus, if M^m is any (closed manifold) homotopy m-sphere, $k > 0$, and $k + m \neq 4$, then $\Sigma^k(M^m) \approx S^{m+k}$. The case $k = 1$, $m = 3$, is lacking a solution.

REMARK 2. The above proofs can easily be modified to yield the following results. If $f: M \rightarrow N$ is an onto map between closed 3-manifolds such that Z_f embeds locally in S^4 then M is homeomorphic to the connected sum $N \# H$, where H is a closed 3-manifold and $\Sigma(H) \approx S^4$. Conversely, if $M = N \# H$ where N and H are closed 3-manifolds and $\Sigma(H) \approx S^4$, then there exists a map of M onto N whose mapping cylinder is a topological manifold with boundary $M \cup N$.

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