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THE NONEXISTENCE OF FREE S^1 ACTIONS ON SOME HOMOTOPY SPHERES

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ABSTRACT. In this paper a necessary condition is given for the existence of a free differentiable action of the circle group S^1 on a $(4k+1)$-dimensional homotopy sphere. This includes a previously known criterion due to R. Lee and yields additional examples of exotic spheres for which no such actions exist.

1. **Statement of results.** Recall that the inertia group $I(M)$ of a closed smooth oriented n-manifold M consists of those homotopy n-spheres Σ for which $M \# \Sigma$ and M are orientation-preservingly diffeomorphic [6].

THEOREM. *If Σ^{4k+1} is a homotopy sphere which admits a free differentiable S^1 action, then $I(\Sigma^{4k+1}\times S^1)=0$.*[1]

COROLLARY (R. LEE [4]). *If Σ^{8k+1} does not bound a spin manifold, then Σ^{8k+1} has no free differentiable S^1 action.*

The Corollary follows because if Σ^{8k+1} is as given, then $I(\Sigma^{8k+1}\times S^1)\neq 0$ by [6, 2.12] or [1].

2. **Examples.** Let $\eta^*\in\Gamma_{16}=Z_2$ and $\eta_n\in\pi_{n+1}(S^n)=Z_2$ $(n\geqq 3)$ be the respective generators. Then by [8, p. 189] and [6, 2.10], the composition $\eta^*\eta_{16}\eta_{17}$ is an element of $\Gamma_{18}=\pi_{18}(PD/O)$ having order 2. The composition $\eta^*\eta_{16}$ corresponds to a homotopy 17-sphere Σ^{17} which bounds a spin manifold [1]. But $\eta^*\eta_{16}\eta_{17}\neq 0$ implies that $I(\Sigma^{17}\times S^1)\neq 0$ (compare [6, p. 197]), and hence Σ^{17} has no free S^1 action. From [5, Table 1.1.7] we may observe that there also exist exotic 33- and 41-spheres which bound spin manifolds but have no free S^1 actions. It seems likely that there is an infinite class of such examples.

The above Theorem contrasts with the fact that if Σ^{4k-1} is in the inertia group of $M^{4k-2}\times S^1$ (M^{4k-2} some homotopy sphere), then Σ^{4k-1} has a free S^1 action. For a nontrivial example in the case $k=4$ see [2, p. 447].

3. **Proof of Theorem.** Let $f:\Sigma\times S^1\to\Sigma$ be the smooth action; then f induces a diffeomorphism $F=(f,\ \pi_2):\Sigma\times S^1\to\Sigma\times S^1$ and a con-

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[1] D. Frank has another proof of the above theorem (obtained independently).

tinuous map $f':S^1\to G_{4k+2}$. (Note: G_{4k+2} is the topological monoid of all self-maps of S^{4k+1} having degree $+1$.) Let $g:S^1\to G_{4k+2}$ be the map which sends a 2×2 matrix in $S^1=SO_2$ into $(2k+1)$ copies down the diagonal of $SO_{4k+2}\subseteq G_{4k+2}$. Then the fact that the orbit space Σ/f has the homotopy type of CP^{2k} [**3**] implies that g and f' are homotopy conjugate in G_{4k+2} and hence homotopic. But the homotopy class $[g]\in\pi_1(G_{4k+2})=Z_2$ is merely $(2k+1)$ times the generator η, which of course is again η.

Now $g_\#:S^{4k+1}\times S^1\to S^{4k+1}\times S^1$ is a diffeomorphism, and hence by the composition rule for smoothings of a PL manifold (compare [**6**, 2.2]), $g_\#^{-1}$ induces a diffeomorphism from some smoothing $g_\#(\Sigma\times S^1)$ to $\Sigma\times S^1$. Explicitly [**6**, 2.4], $g_\#(\Sigma\times S^1)$ is the connected sum $\Sigma\times S^1 \# \Sigma_0^{4k+2}$, where Σ_0^{4k+2} is defined as follows: If $\beta\in\Gamma_{4k+1}=\pi_{4k+1}(PD/O)$ corresponds to Σ^{4k+1}, then the homotopy composition $\beta\circ J([g])=\beta\circ\eta_{4k+1}\in\pi_{4k+2}(PD/O)$ corresponds to Σ_0^{4k+2}.

Let $i:\Sigma\times S^1\to S^{4k+1}\times S^1$ and $i':\Sigma\times S^1 \# \Sigma_0\to S^{4k+2}\times S^1$ be the canonical homeomorphisms. Then the diffeomorphism

$$h = F\circ \overset{-1}{g_\#}:\Sigma\times S^1\#\Sigma_0\to\Sigma\times S^1$$

satisfies $i'\simeq ih$. In other words, $(\Sigma\times S^1 \# \Sigma_0, i')$ and $(\Sigma\times S^1, i)$ are equivalent as homotopy smoothings of $S^{4k+1}\times S^1$. On the other hand, it follows from surgery theory that they must be concordant as topological or combinatorial smoothings. Hence $\Sigma_0^{4k+2}=S^{4k+2}$, and by the previous paragraph $\beta\eta=0$. But according to [**6**, 2.5], this implies that $I(\Sigma^{4k+1}\times S^1)=0$.

4. Semifree actions. The above Theorem parallels a result on semifree S^1 actions (i.e., free on the complement of the fixed point set).

PROPOSITION. *Let Σ^n be a homotopy sphere such that $I(\Sigma^n\times S^1)\neq 0$. If Σ^n has a smooth semifree S^1 action, then the codimension of its fixed point set is divisible by* 4.

The Theorem may be interpreted as treating the case of a (-1)-dimensional fixed point set. Thus in the Proposition we may assume that the action has a fixed point.

PROOF. Let $x\in\Sigma^n$ be a fixed point, and consider the local representation

$$\rho_x:S^1\to \mathrm{Diff}_x\,\Sigma^n\to SO_n.$$

($\mathrm{Diff}_x\Sigma$ is the subgroup of the diffeomorphism group of Σ which leaves x fixed; see [**7**].) Since $I(\Sigma^n\times S^1)\neq 0$, the exact sequence of [**7**] implies that ρ_x is homotopically trivial. But ρ_x is basically q

copies of the inclusion of SO_2 down the diagonal, where $2q$ is the codimension of the fixed point set; if such a map is homotopically trivial, then it is clear that q must be even because $\pi_1(SO) = Z_2$ and the mapping induced by $SO_2 \subseteq SO_n$ is onto.

Let Σ^{17} be the exotic sphere discussed above. Bredon has constructed examples of semifree S^1 actions on Σ^{17} such that the codimension of the fixed point set is any multiple of 4 [2]; a similar result holds for the exotic 33-sphere considered above. Thus the above Proposition contains the best possible general restriction on the codimension of the fixed point set.

References

1. D. W. Anderson, E. H. Brown, Jr. and F. P. Peterson, SU-*cobordism*, KO-*characteristic numbers, and the Kervaire invariant*, Ann. of Math. (2) **83** (1966), 54–67. MR **32** #6470.

2. G. E. Bredon, *A Π_*-module structure for θ_* and applications to transformation groups*, Ann. of Math. (2) **86** (1967), 434–448. MR **36** #4570.

3. W.-C. Hsiang, *A note on free differentiable actions of S^1 and S^3 on homotopy spheres*, Ann. of Math. (2) **83** (1966), 266–272. MR **33** #731.

4. R. Lee, *Non-existence of free differentiable actions of S^1 and Z_2 on homotopy spheres*, Proc. Conference on Transformation Groups (New Orleans, 1967), Springer-Verlag, New York, 1968, pp. 208–209. MR **39** #6352.

5. M. Mahowald and M. Tangora, *Some differentials in the Adams spectral sequence*, Topology 6(1967), 349–369. MR **35** #4924.

6. R. Schultz, *Smooth structures on $S^p \times S^q$*, Ann. of Math. (2) **90** (1969), 187–198.

7. ———, *Improved estimates for the degree of symmetry of certain homotopy spheres*, Topology (to appear).

8. H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies, no. 49, Princeton Univ. Press, Princeton, N. J., 1962. MR **26** #777.

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