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THE ORTHOMODULAR IDENTITY AND METRIC COMPLETENESS OF THE COORDINATIZING DIVISION RING

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ABSTRACT. Let F be any division subring of the real quaternions H. Let $l_2(F)$ denote the linear space of all square summable sequences from F and let L denote the lattice of all "$\perp$-closed" subspaces of $l_2(F)$, where "$\perp$" denotes the orthogonality relation derived from the H-valued form $(a, b) = \sum_{i=1}^{\infty} a_i\bar{b}_i$ where $a, b \in l_2(F)$, $a = (a_i; i = 1, 2, \cdots)$ and $b = (b_i; i = 1, 2, \cdots)$. Then L is complete, orthocomplemented, M-symmetric, irreducible, atomistic, and separable, but L is orthomodular if and only if F is either the reals, the complex numbers, or the quaternions.

The lattice of all closed subspaces of infinite-dimensional, separable, complex Hilbert space has these seven lattice-theoretic properties:

(i) complete [**1**, p. 6];

(ii) orthocomplemented [**1**, p. 52], [**2**, p. 42];

(iii) atomistic (Every element is the join of the atoms beneath it.) [**3**, p. 48];

(iv) irreducible (The center consists precisely of 0 and 1.) [**1**, p. 67], [**3**, p. 27];

(v) separable (An orthogonal family of atoms has at most countably many elements.) and infinite dimensional [**3**, p. 58];

(vi) M-symmetric (If $a, b \in L$, we write aMb if $x \leqq b$ implies $x \vee (a \wedge b) = (x \vee a) \wedge b$. L is M-symmetric if aMb implies bMa.) [**1**, p. 82], [**3**, p. 2];

(vii) orthomodular (If $a, b \in L$ and $a \leqq b$, then $b = a \vee (b \wedge a')$.) [**1**, p. 53], [**2**, p. 42].

Real and quaternionic Hilbert space have the same properties. The question arises whether these are the only three lattices (up to ortho-isomorphism) having them. The problem underlying this question is one of coordinatization, that is, the realization of an abstract lattice, described only by algebraic properties, as a lattice associated in some natural way with a concrete object, for example, the lattice of projections of Hilbert space. Work towards a coordinatization theorem for lattices with the above properties has been done

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by MacLaren [4], [5] and Zierler [6]. The former showed that if L has properties (i) through (vi) and if dim $L \geqq 4$, then L is ortho-isomorphic to the lattice of closed subspaces of a semi-inner-product space over some division ring D. Our question is whether the assumption (vii) is enough to force D to be either the reals, the complex numbers, or the quaternions. An answer of "yes" would characterize completely, in terms of lattice-theoretic properties, projection lattices of Hilbert space and would thus be of great importance in the study of the logical foundations of quantum mechanics [7, p. 71]. We present here some evidence in support of the possibility of an affirmative answer. Let F be any division subring of the real quaternions H. We denote by $|x| = (a^2+b^2+c^2+d^2)^{1/2}$ the norm and by $\bar{x} = a-bi-cj-dk$ the conjugate of $x = a+bi+cj+dk \in H$. The map $x \to \bar{x}$ is an involutory anti-automorphism of H. Consider $l_2(F)$, the linear space of square-summable (with respect to the above norm) sequences from F. Define a definite, Hermitian, conjugate-bilinear "form" on $l_2(F)$, $(\ ,\ ): l_2(F) \times l_2(F) \to H$, by the rule $(x, y) = \sum_{n=1}^{\infty} x_n \bar{y}_n$, where $x, y \in l_2(F)$, $x = (x_n;\ n = 1, 2, \cdots)$ and $y = (y_n;\ n = 1, \cdots)$. Note that this form is H-valued, but not necessarily F-valued. For each subset M of $l_2(F)$, define $M^{\perp} = (y \in l_2(F): (x, y) = 0$ for each $x \in M)$. Call a subspace S of $l_2(F)$ closed in case $S = S^{\perp\perp}$. The map $S \to S^{\perp\perp}$ of the lattice $\bar{L}$ of all subspaces of $l_2(F)$ into itself is a closure operator [8, p. 1518] and so the lattice L of all closed subspaces is complete and orthocomplemented. It is also easily seen that this lattice is irreducible, atomistic, and separable. However:

THEOREM. *L is orthomodular if and only if $F = R$, C, or H.*

PROOF. Only the "only if" part of the theorem needs proof. We give the proof for the case $F \subseteq R$ only (that is, $F \subseteq R$ and L orthomodular imply $F = R$). The proofs of the other two cases (that is, $F \subseteq C$, but $F \not\subseteq R$ and $F \subseteq H$, but $F \not\subseteq C$) follow from the fact that sequential convergence in C or H can be characterized in terms of coordinate-wise convergence in R. Choose $\gamma \in R$. We shall show that, if L is orthomodular, then necessarily $\gamma \in F$. Let $x_0 = 1$ and let $x_n = n/2^n$, $n = 1, 2, 3, \cdots$. Let $x = (x_n;\ n = 0, 1, \cdots)$. Let z_0 be the greatest integer less than or equal to $\gamma \sum_{n=0}^{\infty} x_n^2$. Let $z = (z_n;\ n = 0, 1, \cdots)$ where $.z_1 z_2 \cdots$ is the binary expansion of $\gamma \sum_{n=0}^{\infty} x_n^2 - z_0$. Thus $\sum_{n=1}^{\infty} z_n/2^n = \gamma \sum_{n=0}^{\infty} x_n^2 - z_0$. Let $y_0 = z_0$ and let $y_n = z_n/n$ for $n = 1, 2, \cdots$. Let $y = (y_n;\ n = 0, 1, \cdots)$. Letting $a = \text{sp}(x)$ and $b = \text{sp}(y)$, a and b are distinct atoms in L so that, by orthomodularity, $c = (a \vee b) \wedge a^{\perp} \neq 0$ [3, p. 291]. Necessarily, c is an atom so that $c = \text{sp}(\tau x + y)$ for some $\tau \in F$. But $c \leqq a^{\perp}$ so that $(\tau x + y)$

$\perp x$, that is,

$$0=\sum_{n=0}^{\infty}(\tau x_n+y_n)x_n=\tau\sum_{n=0}^{\infty}x_n^2+\sum_{n=0}^{\infty}x_ny_n.$$

Hence,

$$\tau=-\sum_{n=0}^{\infty}x_ny_n\Big/\sum_{n=0}^{\infty}x_n^2=\left(-z_0-\sum_{n=1}^{\infty}z_n/2^n\right)\left(\sum_{n=0}^{\infty}x_n^2\right)^{-1}$$
$$=-\gamma\left(\sum_{n=0}^{\infty}x_n^2\right)\Big/\sum_{n=0}^{\infty}x_n^2=-\gamma.$$

Since $\tau\in F$, we may conclude $\gamma\in F$, as desired.

ADDED IN PROOF. The fact that $(a\vee b)\wedge a^{\perp}\neq 0$ for distinct atoms a, b also follows from M-symmetry, so the theorem remains valid if we replace "orthomodular" by "M-symmetric." Hence, this L is orthomodular if and only if it is M-symmetric. It follows also that the closure operation $M\rightarrow M^{\perp\perp}$ is Mackey [8, p. 1518] only when $F=R$, C, or H.

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