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ON A PARTITION THEOREM OF MACMAHON-ANDREWS

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ABSTRACT. Two theorems are given about partitions in which the multiplicity of the parts satisfies certain conditions. One of these theorems generalizes a recent result of Andrews concerning partitions in which a part with an odd multiplicity occurs at least $2r+1$ times.

Recently, George Andrews [1] proved the following partition theorem, generalizing an earlier result of MacMahon [2, p. 54] (which deals with the case $r=1$):

The number of partitions of n, in which a part occurring an odd number of times occurs at least $(2r+1)$ times, equals the number of partitions of n into parts which are either even or else $\equiv 2r+1 \pmod{4r+2}$.

We wish to remark that Andrews' theorem is itself a special case of the following result.

THEOREM (A). *Let k be any integer >1 and l any positive integer $\not\equiv 0 \pmod{k}$. Let $A_{k,l}(n)$ be the number partitions of n in which the multiplicity of each part is either $\equiv 0 \pmod{k}$ or else $\geqq l$ and $\equiv l \pmod{k}$. Let $B_{k,l}(n)$ denote the number of partitions of n in which the parts are either $\equiv 0 \pmod{k}$ or else $\equiv l \pmod{2l}$. Then $A_{k,l}(n)=B_{k,l}(n)$.*

Andrews' result corresponds to the choice $k=2$, $l=2r+1$. The proof of this is analogous to that of Andrews' and is therefore omitted.

It is possible to obtain several results of this kind. As a sample, we give the following:

THEOREM B. *Let $m>1$, $r\geqq 0$ be integers, and let $C_{m,r}(n)$ be the number of partitions of n such that all even multiplicities of the parts are less than $2m$, and all odd multiplicities are at least $2r+1$ and at most $2(m+r)-1$. Let $D_{m,r}(n)$ be the number of partitions of n into parts which are either odd and $\equiv 2r+1 \pmod{4r+2}$, or even and $\not\equiv 0 \pmod{2m}$. Then $C_{m,r}(n)=D_{m,r}(n)$.*

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PROOF.

$$1+\sum_{n=1}^{\infty} C_{m,r}(n)x^n = \prod_{n=1}^{\infty}\{1+x^{2n}+x^{4n}+\cdots+x^{(2m-2)n}+x^{(2r+1)n}$$
$$+x^{(2r+3)n}+\cdots+x^{(2m+2r-1)n}\}$$

$$(1)\qquad = \prod_{n=1}^{\infty}(1-x^{2mn})(1-x^{2n})^{-1}(1+x^{(2r+1)n})$$

$$(2)\qquad = \prod_{n=1;\,k\not\equiv 0 \pmod m}^{\infty}(1-x^{2kn})^{-1}\prod_{n=1}^{\infty}(1-x^{(2n-1)(2r+1)})^{-1}$$

$$= 1+\sum_{n=1}^{\infty} D_{m,r}(n)x^n,$$

where we used a well-known Euler identity [2, pp. 10–11], to transform the last product in (1) into the last product in (2). This completes the proof.

The above two theorems can of course be restated using, for the definitions of $A_{k,l}(n)$, $B_{k,l}(n)$, $C_{m,r}(n)$ and $D_{m,r}(n)$, the conjugates of the concerned partitions.

As a particularly interesting special case of the last theorem, we obtain, on taking $m=2$, $r=1$, the following:

COROLLARY. *The number of partitions of n, in which each part occurs two, three or five times, equals the number of partitions of n into parts which are of the forms* 2 (mod 4) *or* 3 (mod 6).

REFERENCES

1. George E. Andrews, *A generalization of a partition theorem of MacMahon*, J. Combinatorial Theory **3** (1967), 100–101. MR **35** #2766.

2. P. A. MacMahon, *Combinatory analysis*, Vol. 2, Reprint Chelsea, New York, 1960. MR **25** #5003.

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