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TOPOLOGICAL INVARIANT MEANS ON LOCALLY COMPACT GROUPS AND FIXED POINTS

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ABSTRACT. A locally compact group G is said to have the fixed point property if whenever G acts affinely on a compact convex subset S of a separated locally convex space E with the map $G\times S\to S$ jointly continuous, there is a fixed point for the action. N. Rickert has proved that G has this fixed point property if G is amenable. In this paper, we study the fixed point property for actions of the algebras $L_1(G)$ and $M(G)$ and prove some fixed point theorems for locally compact groups.

1. **Introduction.** Let G be a locally compact group, G is said to have the fixed point property if whenever G acts affinely on a compact convex subset S in a separated locally convex space E with the map $G\times S\to S$ (denoted by $(x, s)\to T_x(s)$) jointly or separately continuous,[2] there is a point $s_0\in S$ such that $T_x(s_0)=s_0$ for any $x\in G$. N. Rickert [16] has proved that G is amenable iff G has the above fixed point property.

In this paper, we consider (analogous) fixed point properties for actions of the group algebra $L_1(G)$ and the measure algebra $M(G)$ (of all bounded regular Borel measures on G). It turns out that they are both characterisations of amenability of G.

For general terms in harmonic analysis, we follow Hewitt and Ross [10]. This paper is actually a sequel to the author's paper [20] and we shall freely employ the notations and definitions there.

2. **Preliminaries.** Let G be a locally compact group, $L_1(G)$ its group algebra (with convolution as multiplication), $M(G)$ its measure

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[2] R. Ellis has proved in [5] that any such separately continuous action of G is necessarily jointly continuous.

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algebra (of all bounded regular Borel measures on G), E a separated locally convex space. An action of $L_1(G)$ on E is a homomorphism T of $L_1(G)$ into the algebra (under functional composition as multiplication) of all linear operators in E. Thus we have a bilinear mapping $L_1(G)\times E\to E$ denoted by $(\phi, s)\to T_\phi(s)$ such that $T_{\phi*\psi} = T_\phi \circ T_\psi$ for any $\phi, \psi\in L_1(G)$.

Similarly we can define an action $M(G)$ on E. Let $\mathfrak{J}$ be any topology on $L_1(G)$ ($M(G)$), we say that an action T of $L_1(G)$ ($M(G)$) on E is $\mathfrak{J}$-*separately continuous* if the map $L_1(G)\times E\to E$ ($M(g)\times E\to E$) is separately continuous when $L_1(G)$ ($M(G)$) has the topology $\mathfrak{J}$.

Let S be a compact convex subset of E, we say that S is $P(G)$-*invariant* under the action $L_1(G)\times E\to E$ if $T_\phi(S)\subset S$ for any $\phi\in P(G) = \{\phi\in L_1(G): \phi\geqq 0,\ \|\phi\|_1=1\}$. If $M_0(G)$ is the set of probability measures in $M(G)$ ($\mu\in M_0(G)$ iff $\mu\geqq 0$ and $\|\mu\|=1$). We say that S is $M_0(G)$-*invariant* under the action $M(G)\times E\to E$ if $T_\mu(S)\subset S$ for any $\mu\in M_0(G)$. In the first case, T induces an action of the convolution semigroup $P(G)$ on the compact convex subset S (as affine maps now) still denoted by $T:P(G)\times S\to S$. In the second case, T induces an action $T:M_0(G)\times S\to S$ of the convolution semigroup $M_0(G)$ on S. We shall refer to them as the induced actions.

Recall that a linear subspace X of $L_\infty(G)$ is said to be topological left invariant if $\phi * f\in X$ for any $f\in X$ and $\phi\in L_1(G)$. It is said to be topological left introverted if $m_L(f)\in X$ for any $f\in X$ and $m\in L_\infty(G)^*$ where $m_L(f)$ is defined by $m_L(f)(g)=m((1/\Delta)g^\sim * f)$. See [20] for details of these definitions.

3. Fixed point theorems.

THEOREM 3.1. *Let X be a topological left introverted and topological left invariant linear subspace of $L_\infty(G)$ containing the constants. Then the following conditions are equivalent*:

(a) *X has a topological left invariant mean.*[3]

(b) *For any $\sigma(L_1, X)$-separately continuous action $T:L_1(G)\times E\to E$ of $L_1(G)$ on a separated locally convex space E as linear operators in E and any compact convex $P(G)$-invariant subset S of E, the induced action $T:P(G)\times S\to S$ has a fixed point.*

PROOF. Assume that X has a topological left invariant mean, then there is a net $\phi_\alpha\in P(G)$ such that $\phi * \phi_\alpha-\phi_\alpha\to 0$ in $\sigma(L_1, X)$ topology of $L_1(G)$ for any $\phi\in P(G)$.[4] Consider the net $T_{\phi_\alpha}(s)$ where $s\in S$ is

[3] A mean m on X is called topological left invariant if $m(\phi * f) = m(f)$ for any $f\in X$, $\phi\in P(G)$.

[4] For $X=L_\infty(G)$, this is well known. See Greenleaf [9, §2.4]. For general X, this can be proved similarly.

arbitrary but fixed. By compactness of S, we can assume $T_{\phi_\alpha}(s)\to s_0$ in S, passing to a subset if necessary. We claim that s_0 is the required fixed point of $T:P(G)\times S\to S$. For if $\phi\in P(G)$

$$T_\phi(s_0) = T_\phi\left(\lim_\alpha T_{\phi_\alpha}(s)\right) = \lim_\alpha T_\phi(T_{\phi_\alpha}(s)) = \lim_\alpha T_{\phi*\phi_\alpha}(s)$$
$$= \lim_\alpha\{T_{\phi*\phi_\alpha-\phi_\alpha}(s) + T_{\phi_\alpha}(s)\} = \lim_\alpha T_{\phi_\alpha}(s) = s_0,$$

by $\sigma(L_1, X)$-separate continuity of $T:L_1(G)\times E\to E$, linearity of $\phi\to T_\phi(s)$ and the fact that $\phi*\phi_\alpha-\phi_\alpha\to 0$ in $\sigma(L_1, X)$ topology of $L_1(G)$. Hence (a) implies (b).

Conversely, let $E=X^*$ with weak* topology and define $T:L_1(G)\times E\to E$ by $T_\phi(\mu)=l_\phi^*\mu$ (where l_ϕ is defined by $l_\phi f=(1/\Delta)\phi^\sim * f, f\in X$, as in [20]). Since $l_{\phi*\psi}=l_\psi\circ l_\phi$ it is clear that T defines an action of $L_1(G)$ on E (linearity in ϕ or μ is obvious). We claim that T is $\sigma(L_1, X)$-separately continuous. Suppose $\phi_\alpha\to\phi$ in $\sigma(L_1, X)$, then for fixed $\mu\in X^*$ and $f\in X$, we have

$$T_{\phi_\alpha}(\mu)(f) - T_\phi(\mu)(f) = \mu(l_{\phi_\alpha}f) - \mu(l_\phi f) = \mu_L(f)(\phi_\alpha - \phi) \to 0$$

since X is topological left introverted. Therefore $T_{\phi_\alpha}(\mu)\to T_\phi(\mu)$ in E. Evidently the map $\mu\to T_\phi(\mu)$ is continuous for fixed ϕ since l_ϕ^* is $\omega^*-\omega^*$ continuous. Now take S to be the set of all means in X^*. Then S is ω^* compact convex and is $P(G)$-invariant under the $\sigma(L_1, X)$-separately continuous action $T:L_1(G)\times E\to E$ defined above. Consequently, if we assume (b), the induced action $T:P(G)\times S\to S$ must have a fixed point which is a topological left invariant mean on X. Therefore (b) implies (a).

REMARK 3.2. (i) The assumption that X is topological left introverted is not needed in proving (a) implies (b).

(ii) In the proof of the theorem, we have never used the linearity of the map $T:L_1(G)\times E\to E$ where $(\phi, s)\to T_\phi(s)$, in the variable s (for fixed ϕ). This is not surprising since the linearity of the same map in ϕ (for fixed s) and the condition $T_{\phi*\psi}=T_\phi\circ T_\psi$ $(\phi, \psi\in L_1(G))$ together imply that T is linear on the subspace $E_1=\{T_\phi(s):\phi\in L_1(G)\}^-$ of E (for each fixed s). This follows from the continuity of T and the fact that

$$T_\phi(\alpha_1T_{\phi_1}(s) + \alpha_2T_{\phi_2}(s)) = T_\phi(T_{\alpha_1\phi_1+\alpha_2\phi_2}(s)) = T_{\phi*(\alpha_1\phi_1+\alpha_2\phi_2)}(s)$$
$$= T_{\alpha_1\phi*\phi_1+\alpha_2\phi*\phi_2}(s) = \alpha_1T_{\phi*\phi_1}(s) + \alpha_2T_{\phi*\phi_2}(s)$$
$$= \alpha_1T_\phi(T_{\phi_1}(s)) + \alpha_2T_\phi(T_{\phi_2}(s)),$$

for any α_1, α_2 real and $\phi_1, \phi_2\in L_1(G)$. The space E_1 is called the orbit of s.

(iii) Notice that if the action $T:L_1(G)\times E\to E$ is $\sigma(L_1, L_\infty)$ separately continuous, then a fortiori it is separately continuous (i.e. when $L_1(G)$ has the norm topology). The converse might not be true. However, one can eliminate this apparent setback by considering the weak topology in E. Thus if the action $T:L_1(G)\times E\to E$ is separately continuous, then it is also $\sigma(L_1, L_\infty)$-separately continuous when E has the topology $\sigma(E, E^*)$ [**18**, Proposition 13, p. 39].

Consequently fixed point property with respect to $\sigma(L_1, L_\infty)$-separately continuous action and fixed point property with respect to separately continuous action are equivalent (compact subsets of E remain compact in $\sigma(E, E^*)$ topology).

(iv) Theorem 3.1 is "more general" than Rickert's fixed point theorem [**17**, Theorem 4.2, p. 227] in the sense that it applies to any topological left introverted subspace X of $L_\infty(G)$, for example $X=L_\infty(G)$ itself or $X=UCB_r(G)$ the space of all bounded right uniformly continuous functions on G, while Rickert's theorem concerns only $UCB_r(G)$.

We naturally expect Theorem 3.1 to have an analogue for actions of $M(G)$. For general X, there is the difficulty of finding an analogous topology for $M(G)$ corresponding to the topology $\sigma(L_1, X)$ of $L_1(G)$. However, when $X=L_\infty(G)$, we have the following theorem.

THEOREM 3.3. *Let G be a locally compact group, then the following conditions on G are equivalent*:

(a) *G is amenable (i.e. $L_\infty(G)$ has a topological left invariant mean).*

(b) *For any separately continuous action*[5] *$T:L_1(G)\times E\to E$ of $L_1(G)$ on a separated locally convex space E as linear operators in E and any compact convex $P(G)$-invariant subset S of E, the induced action $T:P(G)S\to S$ has a fixed point.*

(c) *For any separately continuous action $T:M(G)\times E\to E$ of $M(G)$ on a separated locally convex space E as linear operators in E and any compact convex $M_0(G)$-invariant subset S of E, the induced action $T:M_0(G)\times S\to S$ has a fixed point.*

PROOF. In view of Theorem 3.2 (when $X=L_\infty(G)$) and Remark 3.3 (iii), it is clear that (a) implies (b).

Next assume (b) and let $T:M(G)\times E\to E$ be any separately continuous action of $M(G)$ on E. Since $L_1(G)$ is a subalgebra of $M(G)$, the restriction $T:L_1(G)\times E\to E$ is a separately continuous action of $L_1(G)$ on E. If S is a compact convex $M_0(G)$-invariant subset of E,

[5] If $\mathfrak{I}$ is the norm topology of $L_1(G)$ or $M(G)$, we use the term separately continuous action for $\mathfrak{I}$-separately continuous action.

then S is clearly $P(G)$-invariant since $P(G)\subset M_0(G)$. Therefore the induced action $T:P(G)\times S\to S$ must have a fixed point (by assumption (b)) which is necessarily a fixed point of $T:M_0(G)\times S\to S$ since $P(G)$ is an ideal of $M_0(G)$ (as convolution semigroup). Hence (b) implies (c).

Finally assume (c) and we want to prove (a). For each $\mu\in M(G)$, define a map $l_\mu:L_\infty(G)\to L_\infty(G)$ by $l_\mu(f)=\mu^{\sim}*f$[6] (see [10, Theorem 20.23]) l_μ is bounded linear [10, Theorem 20.12] and $l_{\mu*\nu}=l_\nu\circ l_\mu$ (a consequence of Fubini's Theorem). Let $E=L_\infty(G)^*$ with weak* topology. Define $T:M(G)\times E\to E$ by $T_\mu(m)=l_\mu^*m$, $\mu\in M(G)$, $m\in L_\infty(G)^*$. Then clearly T is a separately continuous action of $M(G)$ on E. Let S be the set of all means on $L_\infty(G)$. Then S is weak* compact convex and $M_0(G)$-invariant under T. Therefore the induced action $T:M_0(G)\times E\to E$ has a fixed point which is evidently a topological left invariant mean on $L_\infty(G)$. This completes the proof.

4. Silverman's invariant extension property.

DEFINITION 4.1. Let G be a locally compact group, $L_1(G)$ its group algebra, M an abstract M-space with unit e (see [10] for definition). A *right* action of $L_1(G)$ on M is an antihomomorphism of $L_1(G)$ into the algebra of linear operators in M, denoted by $T:L_1(G)\times M\to M$ where $(\phi, s)\to T_\phi(s)$ (this means that $(\phi, s)\to T_\phi(s)$ is bilinear and $T_{\phi*\psi}=T_\psi\circ T_\phi$ for any $\phi, \psi\in L_1(G)$) such that $T_\phi: M\to M$ is a positive linear operator and $T_\phi(e)=e$ for any $\phi\in P(G)$. Suppose $\mathfrak{J}$ is a topology on $L_1(G)$. The right action T is called $\mathfrak{J}$-weakly separately continuous when $L_1(G)$ has the topology $\mathfrak{J}$ and M has the weak topology $\sigma(M, M^*)$ (see also Namioka [10] for his definition of right actions which is slightly different from ours).

If A is a linear subspace of M with $e\in A$, we say that $v\in A^*$ is a mean on A if $\|v\|=v(e)=1$. A is called $P(G)$-invariant under the right action T if $T_\phi(A)\subset A$ for any $\phi\in P(G)$.

Given a $\mathfrak{J}$-weakly separately continuous right action $T:L_1(G)\times M\to M$, consider the separated locally convex space $E=M^*$ with weak* topology $\sigma(M^*, M)$. It is easily verified that the right action T induces an action $T^*:L_1(G)\times E\to E$ (in the sense of §3) with $(\phi, \mu)\to T_\phi^*\mu$ for any $\phi\in L_1(G)$, $\mu\in M^*$. Moreover the induced action is $\mathfrak{J}$-separately continuous (note that since each $T_\phi: M\to M$ is weak-weak continuous, it is continuous and hence $T_\phi^*: M\to M^*$ is well defined).

THEOREM 4.2. *Let X be a topological left introverted and topological left invariant linear subspace of $L_\infty(G)$ containing the constants. Sup-*

[6] $\mu^{\sim}$ is defined by $\mu^{\sim}(E)=\mu(E^{-1})$ for any Borel set E in G.

pose X has a topological left invariant mean. Let $T: L_1(G) \times M \to M$ be a $\sigma(L_1, X)$-weakly separately continuous right action of $L_1(G)$ on an abstract M-space M with unit e and A any $P(G)$-invariant subspace containing e. If $\nu \in A^*$ is a mean on A such that $\nu(T_\phi(s)) = \nu(s)$ for any $\phi \in P(G)$, $s \in A$, then there is a mean $\mu \in M^*$ extending ν such that $\mu(T_\phi(s)) = \mu(s)$ for any $\phi \in P(G)$ and $s \in M$.

Conversely, if we assume, in addition, that X is an abstract M space (e.g. $X = L_\infty(G)$), then X admits a topological left invariant mean if X has the above "invariant extension property."

PROOF. Consider the induced action $L_1(G) \times M^* \to M^*$ which sends (ϕ, μ) to $T_\phi^*\mu$. As remarked before, this action is $\sigma(L_1, X)$-separately continuous when M^* has the weak* topology. Let $S = \{\mu \in M^* : \mu$ is a mean on M and μ extends $\nu\}$. $S \neq \emptyset$ by the Hahn-Banach Theorem. In fact S is a weak* closed convex subset of the unit ball in M^* and is therefore weak* compact. We claim that $T_\phi(S) \subset S$ for any $\phi \in P(G)$. Let $\mu \in S$, $\phi \in P(G)$. Since $T_\phi : M \to M$ is positive linear and $T_\phi(e) = e$, we have $\|T_\phi(s)\| \leqq \|s\|$ for any $s \in M$. Therefore $\|T_\phi\| \leqq 1$ and $\|T_\phi^*\mu\| \leqq \|T_\phi\| \cdot \|\mu\| \leqq \|\mu\| = 1$. Consequently $\|T_\phi^*\mu\| = T_\phi^*\mu(e) = 1$ or $T_\phi^*\mu$ is a mean on M. Also $T_\phi^*\mu$ extends ν since if $s \in A$, then $T_\phi^*\mu(s) = \mu(T_\phi(s)) = \nu(T_\phi(s)) = \nu(s)$ (recall that $T_\phi(A) \subset A$). Therefore S is $P(G)$-invariant under the action $T^* : L_1(G) \times M \to M^*$. By Theorem 3.2, there is some $\mu \in S$ such that $T_\phi^*\mu = \mu$ for any $\phi \in P(G)$. μ is then the required extension of ν.

Conversely, assume that X is also an abstract M-space and that X has this "invariant extension property," we can take $M = X$ and define a right action $L_1(G) \times M \to M$ by $T_\phi(f) = l_\phi f = (1/\Delta)\phi^\sim * f$, $f \in X$ and $\phi \in L_1(G)$. We claim that it is $\sigma(L_1, X)$-weakly separately continuous. For if $\phi_\alpha \to \phi$ in $\sigma(L_1, X)$, then for any $\mu \in X^*, f \in X$,

$$\mu(T_{\phi_\alpha}(f) - T_\phi(f)) = \mu\left(\frac{1}{\Delta}\tilde{\phi_\alpha} * f - \frac{1}{\Delta}\tilde{\phi} * f\right) = \mu_L(f)(\phi_\alpha - \phi) \to 0.$$

($\mu_L(f) \in X$ since X is topological left introverted.) Hence $T_{\phi_\alpha}(f) \to T_\phi(f)$ weakly in M. On the other hand, it is clear that each $T_\phi : M \to M$ is continuous. Now choose A to be the constants and define $\nu(\beta \cdot 1) = \beta$ for any $\beta \cdot 1 \in A$. Then A is obviously $P(G)$-invariant under the $\sigma(L_1, X)$-weakly separately continuous right action $T: L_1(G) \times M \to M$ and ν is a mean on A satisfying $\nu(T_\phi(f)) = \nu(f)$ for any $\phi \in P(G)$ and $f \in A$. Any invariant extension μ of ν to $M = X$ is necessarily a topological left invariant mean on X.

REMARK 4.3. (i) The above characterisation of amenability of G (i.e. when $X = L_\infty(G)$) is an analogue of a theorem of R. J. Silverman [**19**, Theorem 15] for amenable semigroups.

(ii) Right actions of $M(G)$ on an abstract M-space with unit can also be defined and a theorem of similar type to Theorem 4.2 (with $X=L_\infty(G)$) can also be proved with obvious modifications.

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