TOPOLOGICAL INVARIANT MEANS ON LOCALLY
COMPACT GROUPS AND FIXED POINTS

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Abstract. A locally compact group $G$ is said to have the fixed point property if whenever $G$ acts affinely on a compact convex subset $S$ of a separated locally convex space $E$ with the map $G \times S \to S$ jointly or separately continuous, there is a fixed point for the action. N. Rickert has proved that $G$ has this fixed point property if $G$ is amenable. In this paper, we study the fixed point property for actions of the algebras $L_1(G)$ and $M(G)$ and prove some fixed point theorems for locally compact groups.

1. Introduction. Let $G$ be a locally compact group, $G$ is said to have the fixed point property if whenever $G$ acts affinely on a compact convex subset $S$ in a separated locally convex space $E$ with the map $G \times S \to S$ (denoted by $(x, s) \to T_x(s)$) jointly or separately continuous, there is a point $s_0 \in S$ such that $T_x(s_0) = s_0$ for any $x \in G$. N. Rickert [16] has proved that $G$ is amenable iff $G$ has the above fixed point property.

In this paper, we consider (analogous) fixed point properties for actions of the group algebra $L_1(G)$ and the measure algebra $M(G)$ (of all bounded regular Borel measures on $G$). It turns out that they are both characterisations of amenability of $G$.

For general terms in harmonic analysis, we follow Hewitt and Ross [10]. This paper is actually a sequel to the author's paper [20] and we shall freely employ the notations and definitions there.

2. Preliminaries. Let $G$ be a locally compact group, $L_1(G)$ its group algebra (with convolution as multiplication), $M(G)$ its measure
algebra (of all bounded regular Borel measures on $G$), $E$ a separated locally convex space. An action of $L_1(G)$ on $E$ is a homomorphism $T$ of $L_1(G)$ into the algebra (under functional composition as multiplication) of all linear operators in $E$. Thus we have a bilinear mapping $L_1(G) \times E \to E$ denoted by $(\phi, s) \mapsto T_\phi(s)$ such that $T_{\phi \psi} = T_\phi \circ T_\psi$ for any $\phi, \psi \in L_1(G)$.

Similarly we can define an action $M(G)$ on $E$. Let $\tau$ be any topology on $L_1(G)$ ($M(G)$), we say that an action $T$ of $L_1(G)$ ($M(G)$) on $E$ is $\tau$-separately continuous if the map $L_1(G) \times E \to E$ ($M(G) \times E \to E$) is separately continuous when $L_1(G)$ ($M(G)$) has the topology $\tau$.

Let $S$ be a compact convex subset of $E$, we say that $S$ is $\tau$-invariant under the action $T$ if $T_\phi(S) \subseteq S$ for any $\phi \in \tau$. If $M_0(G)$ is the set of probability measures in $M(G)$ ($\mu \in M_0(G)$ iff $\mu \geq 0$ and $\|\mu\| = 1$). We say that $S$ is $M_0(G)$-invariant under the action $M(G) \times E \to E$ if $T_\mu(S) \subseteq S$ for any $\mu \in M_0(G)$. In the first case, $T$ induces an action of the convolution semigroup $P(G)$ on the compact convex subset $S$ (as affine maps now) still denoted by $T: P(G) \times S \to S$. In the second case, $T$ induces an action $T: M_0(G) \times S \to S$ of the convolution semigroup $M_0(G)$ on $S$. We shall refer to them as the induced actions.

Recall that a linear subspace $X$ of $L_\infty(G)$ is said to be topological left invariant if $\phi \ast f \in X$ for any $f \in X$ and $\phi \in L_1(G)$. It is said to be topological left introverted if $\phi \ast l(f) \in X$ for any $f \in X$ and $\phi \in L_1(G)$, where $\phi \ast l(f)$ is defined by $\phi \ast l(f)(g) = \mu((1/\Delta)g^{-1} \ast f)$. See [20] for details of these definitions.

3. Fixed point theorems.

**Theorem 3.1.** Let $X$ be a topological left introverted and topological left invariant linear subspace of $L_\infty(G)$ containing the constants. Then the following conditions are equivalent:

(a) $X$ has a topological left invariant mean.

(b) For any $\sigma(L_1, X)$-separately continuous action $T: L_1(G) \times E \to E$ of $L_1(G)$ on a separated locally convex space $E$ as linear operators in $E$ and any compact convex $P(G)$-invariant subset $S$ of $E$, the induced action $T: P(G) \times S \to S$ has a fixed point.

**Proof.** Assume that $X$ has a topological left invariant mean, then there is a net $\phi_a \in P(G)$ such that $\phi \ast \phi_a \to 0$ in $\sigma(L_1, X)$ topology of $L_1(G)$ for any $\phi \in P(G)$.

Consider the net $T_s(s)$ where $s \in S$ is

\footnote{A mean $m$ on $X$ is called topological left invariant if $m(\phi \ast f) = m(f)$ for any $f \in X$, $\phi \in P(G)$.}

\footnote{For $X = L_\infty(G)$, this is well known. See Greenleaf [9, §2.4]. For general $X$, this can be proved similarly.}
arbitrary but fixed. By compactness of $S$, we can assume $T_{\phi}(s) \rightarrow s_0$ in $S$, passing to a subset if necessary. We claim that $s_0$ is the required fixed point of $T: P(G) \times S \rightarrow S$. For if $\phi \in P(G)$

$$T_{\phi}(s_0) = T_{\phi}\left(\lim_{\alpha} T_{\phi}(s)\right) = \lim_{\alpha} T_{\phi}(T_{\phi}(s)) = \lim_{\alpha} T_{\phi}(s)$$

$$= \lim_{\alpha} \{T_{\phi}(s_0) + T_{\phi}(s)\} = \lim_{\alpha} T_{\phi}(s) = s_0,$$

by $\sigma(L_1, X)$-separate continuity of $T: L_1(G) \times E \rightarrow E$, linearity of $\phi \rightarrow T_{\phi}(s)$ and the fact that $\phi * \phi_a - \phi_a \rightarrow 0$ in $\sigma(L_1, X)$ topology of $L_1(G)$. Hence (a) implies (b).

Conversely, let $E = X^*$ with weak* topology and define $T: L_1(G) \times E \rightarrow E$ by $T_{\phi}(\mu) = l_{\phi}^* \mu$ (where $l_{\phi}$ is defined by $l_{\phi}f = (1/\Delta)\phi^* f$, $f \in X$, as in [20]). Since $l_{\phi, \psi} = l_{\phi} \circ l_{\psi}$ it is clear that $T$ defines an action of $L_1(G)$ on $E$ (linearity in $\phi$ or $\mu$ is obvious). We claim that $T$ is $\sigma(L_1, X)$-separately continuous. Suppose $\phi_a \rightarrow \phi$ in $\sigma(L_1, X)$, then for fixed $\mu \in X^*$ and $f \in X$, we have

$$T_{\phi_a}(\mu)(f) - T_{\phi}(\mu)(f) = \mu(l_{\phi_a}f) - \mu(l_{\phi}f) = \mu_L(f)(\phi_a - \phi) \rightarrow 0$$

since $X$ is topological left introverted. Therefore $T_{\phi_a}(\mu) \rightarrow T_{\phi}(\mu)$ in $E$. Evidently the map $\mu \rightarrow T_{\phi}(\mu)$ is continuous for fixed $\phi$ since $l_{\phi}^*$ is $\omega^* - \omega^*$ continuous. Now take $S$ to be the set of all means in $X^*$. Then $S$ is $\omega^*$ compact convex and is $P(G)$-invariant under the $\sigma(L_1, X)$-separately continuous action $T: L_1(G) \times E \rightarrow E$ defined above. Consequently, if we assume (b), the induced action $T: P(G) \times S \rightarrow S$ must have a fixed point which is a topological left invariant mean on $X$. Therefore (b) implies (a).

Remark 3.2. (i) The assumption that $X$ is topological left introverted is not needed in proving (a) implies (b).

(ii) In the proof of the theorem, we have never used the linearity of the map $T: L_1(G) \times E \rightarrow E$ where $(\phi, s) \rightarrow T_{\phi}(s)$, in the variable $s$ (for fixed $\phi$). This is not surprising since the linearity of the same map in $\phi$ (for fixed $s$) and the condition $T_{\phi, \psi} = T_{\phi} \circ T_{\psi}$ ($\phi, \psi \in L_1(G)$) together imply that $T$ is linear on the subspace $E_1 = \{T_{\phi}(s): \phi \in L_1(G)\}$ of $E$ (for each fixed $s$). This follows from the continuity of $T$ and the fact that

$$T_{\phi}(\alpha_1 T_{\phi_1}(s) + \alpha_2 T_{\phi_2}(s)) = T_{\phi}(\alpha_1 T_{\phi_1}(s) + \alpha_2 T_{\phi_2}(s)) = \alpha_1 T_{\phi}(\alpha_1 T_{\phi_1}(s) + \alpha_2 T_{\phi_2}(s))$$

$$= \alpha_1 T_{\phi}(T_{\phi_1}(s)) + \alpha_2 T_{\phi}(T_{\phi_2}(s)),$$

for any $\alpha_1, \alpha_2$ real and $\phi_1, \phi_2 \in L_1(G)$. The space $E_1$ is called the orbit of $s$. 
(iii) Notice that if the action $T: L_1(G) \times E \to E$ is $\sigma(L_1, L_\omega)$ separately continuous, then a fortiori it is separately continuous (i.e. when $L_1(G)$ has the norm topology). The converse might not be true. However, one can eliminate this apparent setback by considering the weak topology in $E$. Thus if the action $T: L_1(G) \times E \to E$ is separately continuous, then it is also $\sigma(L_1, L_\omega)$-separately continuous when $E$ has the topology $\sigma(E, E^*)$ [18, Proposition 13, p. 39].

Consequently fixed point property with respect to $\sigma(L_1, L_\omega)$-separately continuous action and fixed point property with respect to separately continuous action are equivalent (compact subsets of $E$ remain compact in $\sigma(E, E^*)$ topology).

(iv) Theorem 3.1 is “more general” than Rickert’s fixed point theorem [17, Theorem 4.2, p. 227] in the sense that it applies to any topological left introverted subspace $X$ of $L_\omega(G)$, for example $X = L_\omega(G)$ itself or $X = UCB_r(G)$ the space of all bounded right uniformly continuous functions on $G$, while Rickert’s theorem concerns only $UCB_r(G)$.

We naturally expect Theorem 3.1 to have an analogue for actions of $M(G)$. For general $X$, there is the difficulty of finding an analogous topology for $M(G)$ corresponding to the topology $\sigma(L_1, X)$ of $L_1(G)$. However, when $X = L_\omega(G)$, we have the following theorem.

**Theorem 3.3.** Let $G$ be a locally compact group, then the following conditions on $G$ are equivalent:

(a) $G$ is amenable (i.e. $L_\omega(G)$ has a topological left invariant mean).

(b) For any separately continuous action $T: L_1(G) \times E \to E$ of $L_1(G)$ on a separated locally convex space $E$ as linear operators in $E$ and any compact convex $P(G)$-invariant subset $S$ of $E$, the induced action $T: P(G)S \to S$ has a fixed point.

(c) For any separately continuous action $T: M(G) \times E \to E$ of $M(G)$ on a separated locally convex space $E$ as linear operators in $E$ and any compact convex $M_0(G)$-invariant subset $S$ of $E$, the induced action $T: M_0(G) \times S \to S$ has a fixed point.

**Proof.** In view of Theorem 3.2 (when $X = L_\omega(G)$) and Remark 3.3 (iii), it is clear that (a) implies (b).

Next assume (b) and let $T: M(G) \times E \to E$ be any separately continuous action of $M(G)$ on $E$. Since $L_1(G)$ is a subalgebra of $M(G)$, the restriction $T: L_1(G) \times E \to E$ is a separately continuous action of $L_1(G)$ on $E$. If $S$ is a compact convex $M_0(G)$-invariant subset of $E$,
then $S$ is clearly $P(G)$-invariant since $P(G) \subseteq M_0(G)$. Therefore the induced action $T: P(G) \times S \to S$ must have a fixed point (by assumption (b)) which is necessarily a fixed point of $T: M_0(G) \times S \to S$ since $P(G)$ is an ideal of $M_0(G)$ (as convolution semigroup). Hence (b) implies (c).

Finally assume (c) and we want to prove (a). For each $\mu \in M(G)$, define a map $l_\mu : L_\infty(G) \to L_\infty(G)$ by $l_\mu(f) = \mu \ast f^0$ (see [10, Theorem 20.23]) $l_\mu$ is bounded linear [10, Theorem 20.12] and $l_{\mu \ast} = l_\mu \circ l_\mu$ (a consequence of Fubini’s Theorem). Let $E = L_\infty(G)^*$ with weak* topology. Define $T: M(G) \times E \to E$ by $T_\mu(m) = l_\mu(m)$, $\mu \in M(G)$, $m \in L_\infty(G)^*$. Then clearly $T$ is a separately continuous action of $M(G)$ on $E$. Let $S$ be the set of all means on $L_\infty(G)$. Then $S$ is weak* compact convex and $M_0(G)$-invariant under $T$. Therefore the induced action $T: M_0(G) \times E \to E$ has a fixed point which is evidently a topological left invariant mean on $L_\infty(G)$. This completes the proof.

4. Silverman’s invariant extension property.

**Definition 4.1.** Let $G$ be a locally compact group, $L_1(G)$ its group algebra, $M$ an abstract $M$-space with unit $e$ (see [10] for definition). A right action of $L_1(G)$ on $M$ is an antihomomorphism of $L_1(G)$ into the algebra of linear operators in $M$, denoted by $T: L_1(G) \times M \to M$ where $(\phi, s) \mapsto T_\phi(s)$ (this means that $(\phi, s) \mapsto T_\phi(s)$ is bilinear and $T_{\phi \psi} = T_\psi \circ T_\phi$ for any $\phi, \psi \in L_1(G)$) such that $T_\phi: M \to M$ is a positive linear operator and $T_\phi(e) = e$ for any $\phi \in P(G)$. Suppose $\tau$ is a topology on $L_1(G)$. The right action $T$ is called $\tau$-weakly separately continuous when $L_1(G)$ has the topology $\tau$ and $M$ has the weak topology $\sigma(M, M^*)$ (see also Namioka [10] for his definition of right actions which is slightly different from ours).

If $A$ is a linear subspace of $M$ with $e \in A$, we say that $v \in A^*$ is a mean on $A$ if $\|v\| = v(e) = 1$. $A$ is called $P(G)$-invariant under the right action $T$ if $T_\phi(A) \subseteq A$ for any $\phi \in P(G)$.

Given a $\tau$-weakly separately continuous right action $T: L_1(G) \times M \to M$, consider the separated locally convex space $E = M^*$ with weak* topology $\sigma(M^*, M)$. It is easily verified that the right action $T$ induces an action $T^*: L_1(G) \times E \to E$ (in the sense of §3) with $(\phi, \mu) \mapsto T_\phi^* \mu$ for any $\phi \in L_1(G)$, $\mu \in M^*$. Moreover the induced action is $\tau$-separately continuous (note that since each $T_\phi: M \to M$ is weak-weak continuous, it is continuous and hence $T_\phi^*: M \to M^*$ is well defined).

**Theorem 4.2.** Let $X$ be a topological left introverted and topological left invariant linear subspace of $L_\infty(G)$ containing the constants. Sup-

\[ * \] $\mu^*$ is defined by $\mu^*(E) = \mu(E^{-1})$ for any Borel set $E$ in $G$. 
pose $X$ has a topological left invariant mean. Let $T:L_1(G) \times M \to M$ be a $\sigma(L_1, X)$-weakly separately continuous right action of $L_1(G)$ on an abstract $M$-space $M$ with unit $e$ and $A$ any $P(G)$-invariant subspace containing $e$. If $v \in A^*$ is a mean on $A$ such that $v(T_\phi(s)) = v(s)$ for any $\phi \in P(G)$, $s \in A$, then there is a mean $\mu \in M^*$ extending $v$ such that $\mu(T_\phi(s)) = \mu(s)$ for any $\phi \in P(G)$ and $s \in M$.

Conversely, if we assume, in addition, that $X$ is an abstract $M$-space (e.g. $X = L_\infty(G)$), then $X$ admits a topological left invariant mean if $X$ has the above “invariant extension property.”

Proof. Consider the induced action $L_1(G) \times M^* \to M^*$ which sends $(\phi, \mu)$ to $T_\phi^*\mu$. As remarked before, this action is $\sigma(L_1, X)$-separately continuous when $M^*$ has the weak* topology. Let $S = \{\mu \in M^*: \mu$ is a mean on $M$ and $\mu$ extends $v\}$. $S \neq \emptyset$ by the Hahn-Banach Theorem. In fact $S$ is a weak* closed convex subset of the unit ball in $M^*$ and is therefore weak* compact. We claim that $T_\phi(S) \subseteq S$ for any $\phi \in P(G)$. Let $\mu \in S$, $\phi \in P(G)$. Since $T_\phi:M \to M$ is positive linear and $T_\phi(e) = e$, we have $\|T_\phi(s)\| \leq \|s\|$ for any $s \in M$. Therefore $\|T_\phi\| \leq 1$ and $\|T_\phi^*\mu\| \leq \|\mu\| \leq \|\mu\| = 1$. Consequently $\|T_\phi^*\mu\| = T_\phi^*\mu(e) = 1$ or $T_\phi^*\mu$ is a mean on $M$. Also $T_\phi^*\mu$ extends $v$ since if $s \in A$, then $T_\phi^*\mu(s) = \mu(T_\phi(s)) = v(T_\phi(s)) = v(s)$ (recall that $T_\phi(A) \subseteq A$). Therefore $S$ is $P(G)$-invariant under the action $T^*:L_1(G) \times M \to M^*$. By Theorem 3.2, there is some $\mu \in S$ such that $T_\phi^*\mu = \mu$ for any $\phi \in P(G)$. $\mu$ is then the required extension of $v$.

Conversely, assume that $X$ is also an abstract $M$-space and that $X$ has this “invariant extension property,” we can take $M = X$ and define a right action $L_1(G) \times M \to M$ by $T_\phi(f) = l_\phi f = (1/\Delta) \phi^* f$, $f \in X$ and $\phi \in L_1(G)$. We claim that it is $\sigma(L_1, X)$-weakly separately continuous. For if $\phi_a \to \phi$ in $\sigma(L_1, X)$, then for any $\mu \in X^*$, $f \in X$,

$$
\mu(T_\phi(f) - T_{\phi_a}(f)) = \mu \left( \frac{1}{\Delta} \phi_a^* f - \frac{1}{\Delta} \phi^* f \right) = \mu_L(f)(\phi_a - \phi) \to 0.
$$

($\mu_L(f) \in X$ since $X$ is topological left introverted.) Hence $T_\phi(f) \to_{T_{\phi_a}(f)}$ weakly in $M$. On the other hand, it is clear that each $T_\phi:M \to M$ is continuous. Now choose $A$ to be the constants and define $\nu(\beta \cdot 1) = \beta$ for any $\beta \cdot 1 \in A$. Then $A$ is obviously $P(G)$-invariant under the $\sigma(L_1, X)$-weakly separately continuous right action $T:L_1(G) \times M \to M$ and $\nu$ is a mean on $A$ satisfying $\nu(T_\phi(f)) = \nu(f)$ for any $\phi \in P(G)$ and $f \in A$. Any invariant extension $\mu$ of $\nu$ to $M = X$ is necessarily a topological left invariant mean on $X$.

Remark 4.3. (i) The above characterisation of amenability of $G$ (i.e. when $X = L_\infty(G)$) is an analogue of a theorem of R. J. Silverman [19, Theorem 15] for amenable semigroups.
(ii) Right actions of $M(G)$ on an abstract $M$-space with unit can also be defined and a theorem of similar type to Theorem 4.2 (with $X = L_\infty(G)$) can also be proved with obvious modifications.

References

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