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ZERO SETS OF FUNCTIONS FROM NON-QUASI-ANALYTIC CLASSES

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ABSTRACT. It is well known that any closed subset of the line is the zero set of a C^∞-function. One can also specify the orders of the zeros at the isolated points. The present paper improves this result by replacing the class of C^∞-functions by any non-quasi-analytic class of C^∞-functions.

If $\{M_n\}_{n=0}^\infty$ is a sequence of positive numbers we let $C\{M_n\}$ denote the set of functions f in $C^\infty(R)$ to which there correspond β_f and B_f satisfying

$$\|f^{(n)}\|_\infty \leqq \beta_f B_f^n M_n, \qquad n = 0, 1, \cdots.$$

The purpose of this paper is to prove the following:

THEOREM. *Let $\{M_n\}_{n=0}^\infty$ be a sequence of positive numbers such that $\sum_{n=1}^\infty M_{n-1}/M_n < \infty$. Let E be a closed set in R and let S be a set consisting of at most countably many isolated points of E. Let d be a function which assigns a positive integer to each point in S. Then there is a function f in $C\{M_n\}$ with $\{x \in R : f(x) = 0\} = E$ and furthermore for every s in S the order of the zero of f at s is $d(s)$.*

We let S contain only isolated points since any limit point of the zero set of f could not be a zero of finite order for f. The Denjoy-Carleman Theorem [2, p. 376] shows that a condition such as $\sum_{n=1}^\infty M_{n-1}/M_n < \infty$ is necessary to prevent $C\{M_n\}$ from being quasi-analytic.

We will repeatedly use the following theorem which can be found in [1, pp. 79–84] where it is credited to H. Bray:

THEOREM. *Assume $\{N_n\}_{n=0}^\infty$ is a sequence of positive numbers such that $N_0 = 1$ and $\sum_{n=1}^\infty \lambda_n < \infty$ where $\lambda_n = N_{n-1}/N_n$. Assume g_0 is a bounded measurable function on R which vanishes outside a compact set. For $n = 1, 2, \cdots$ define g_n on R by*

$$g_n(x) = \frac{1}{2\lambda_n} \int_{-\lambda_n}^{\lambda_n} g_{n-1}(x + t)\,dt.$$

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Then $\{g_n\}$ *converges uniformly to a function* g *in* $C^\infty(R)$ *with* $\|g^{(n)}\|_\infty \leq \|g_0\|_\infty N_n$ *for* $n=0, 1, \cdots$.

We will first obtain some functions which will be used in building the function of our theorem. We let $\{s_n\}$ be a strictly increasing sequence of positive numbers satisfying: $s_1=1$, s_n tends to ∞, and $\sum_{n=1}^\infty M_{n-1}s_n/M_n<\infty$. For example we could take

$$s_n = (\text{const})\left(\sum_{k=n}^\infty M_{k-1}/M_k\right)^{-1/2} \qquad \text{for } n > 1.$$

We define $\{N_n\}$ as follows: $N_0=1$ and $N_n=M_n/(s_1 \cdots s_n)$ for $n=1, 2, \cdots$. Then $\sum_{n=1}^\infty N_{n-1}/N_n=\sum_{n=1}^\infty M_{n-1}s_n/M_n<\infty$ and we let λ denote this sum. By applying Bray's theorem to the function which is 1 on $(-\lambda, \lambda)$ and 0 elsewhere we obtain a function g in $C^\infty(R)$ which satisfies:

(i) $0\leq g\leq 1$;

(ii) $g>0$ on $(-2\lambda, 2\lambda)$ and 0 elsewhere; and

(iii) $\|g^{(n)}\|_\infty \leq N_n$ for $n=0, 1, \cdots$.

Scaled translates of g, i.e. functions of the form $Ag(a(t-b))$, will be used to define f in complementary intervals of E whose endpoints do not belong to S.

In order to define f in a complementary interval of E which has at least one endpoint in S we will use the following:

LEMMA. *Let k be a positive integer. Then there are functions $h_1(t, k)$ and $h_2(t, k)$ in $C^\infty(R)$ such that*

(i) $0\leq |h_i| \leq 1$, $i=1, 2$;

(ii) $h_i\neq 0$ *on* $(0, 4\lambda)$ *and* 0 *elsewhere*, $i=1, 2$;

(iii) *there is a number* $c>0$ *such that* $h_1(t, k)=ct^k$ *on* $[0, \lambda]$ *and* $h_2(t, k)=c(t-4\lambda)^k$ *on* $[3\lambda, 4\lambda]$; *and*

(iv) $\|h_i^{(n)}\| \leq N_n$ *for* $n=0, 1, \cdots$ *and* $i=1, 2$.

PROOF. We first observe that if $P(x)$ is a polynomial and $\mu>0$ then

$$\frac{1}{2\mu}\int_{-\mu}^{\mu} P(x+t)dt$$

is a polynomial having the same leading term as $P(x)$. If for each n we apply Bray's process to the function which is x^n on $[-\lambda, 2\lambda]$ and 0 elsewhere, we obtain functions $R_n(x)$ which on $[0, \lambda]$ are polynomials with leading term x^n. Determining coefficients a_i such that on $[0, \lambda]$, $R_k(x)+a_{k-1}R_{k-1}(x)+ \cdots +a_0R_0(x)=x^k$, we obtain a polynomial $x^k+a_{k-1}x^{k-1}+ \cdots +a_0=Q(x)$ such that applying Bray's process to the function which is $Q(x)$ on $[-\lambda, 2\lambda]$ and 0 elsewhere yields a function which on $[0, \lambda]$ is x^k. Let $c>0$ be sufficiently small that $|cQ(x)| \leqq 1$ on $[-\lambda, 3\lambda]$. Let h be the function in $C^\infty(R)$ obtained by applying Bray's process to the function which is $cQ(x)$ on $[-\lambda, 2\lambda]$, 1 on $[2\lambda, 3\lambda]$, and 0 elsewhere. We obtain h_1 from h by changing the definition of h to be 0 on $(-\infty, 0]$. h_2 is obtained in a similar way.

We will use scaled translates of g, h_1, h_2 to define f in the complementary intervals of E. We now introduce a function which will be used as a factor to decrease these functions on small complementary intervals. We define $h>0$ on $(0, \infty)$ by $h(t)=1$ for t in $[s_1^{-1}, \infty)$ and $h(t)=(s_1 \cdots s_{n-1})s_n^{-n+1}$ for t in $[s_n^{-1}, s_{n-1}^{-1})$, $n=2, 3, \cdots$. There are exactly two properties of h which we will use. If k is a nonnegative integer, then

(1) $\lim_{t\to 0^+} h(t)t^{-k} = \lim_{n\to\infty} h(s_n^{-1})s_n^k = 0$; and

(2) $\sup_{t>0} h(t)t^{-k} = \sup_{n>0} h(s_n^{-1})s_n^k = s_1 s_2 \cdots s_k$.

We choose a function σ on E which is 0 on $E\backslash S$ and which takes the values $+1$ and -1 on S in such a way that it possesses the following property: assume s and t are in S and s is the largest number in S which is smaller than t; then if $d(t)$ is odd, $\sigma(s)$ and $\sigma(t)$ have opposite signs, while if $d(t)$ is even then $\sigma(s)$ and $\sigma(t)$ have the same sign. The function σ will be used to insure that the function we are building does not vanish in any complementary interval of E.

We now define f. We treat the case where the complement of E has no unbounded components since the other case requires only an easy modification. We let f be 0 on E and write the complement of E as $\bigcup(a_n, b_n)$ where each (a_n, b_n) is a component of the complement of E. On (a_n, b_n) we define

$$\begin{aligned} f(t) = h(b_n - a_n)\{&g(4\lambda[b_n - a_n]^{-1}[t - (a_n + b_n)/2]) \\ &\cdot(1 - |\sigma(a_n)|)(1 - |\sigma(b_n)|) \\ &+ \sigma(a_n)h_1(4\lambda[b_n - a_n]^{-1}[t - a_n], d(a_n)) \\ &+ \sigma(b_n)h_2(4\lambda[b_n - a_n]^{-1}[t - a_n], d(b_n))\}. \end{aligned}$$

We let D be the union of the complement of E, the interior of E, and the set of isolated points of E. Every point of D has a neighborhood on which f is C^∞. Also one checks that for s in S the order of the zero of f at s is $d(s)$ as desired.

We will now show f is continuous on R. For t in a component interval of length l of the complement of E and $n=0, 1, \cdots$ we have

$$(*) \qquad |f^{(n)}(t)| \leqq 2h(l)l^{-n}(4\lambda)^n N_n$$

which tends to 0 with l. Continuity of f off D follows easily from (*) with $n=0$.

We next show that f is differentiable and also that $f'(t)$ is 0 for t in $E\backslash S$. It is easily seen that if $x<y$ then there is a point t_{xy} in $(x, y) \cap D$ such that $f(x)-f(y)=f'(t_{xy})(x-y)$. Assume t_0 is in $E\backslash S$ and let $\epsilon>0$. Using (*) with $n=1$ we see there is a number $c>t_0$ such that $|f'(t)|<\epsilon$ for t in $(t_0, c)\cap D$. Thus for $t_0<s<t<c$ we have

$$|(f(s)-f(t))/(s-t)| = |f'(t_{st})| < \epsilon.$$

Letting $s\to t_0$ we have $|f(t)/(t-t_0)|<\epsilon$ for $t_0<t<c$. Hence $\limsup_{t\to t_0^+} |f(t)/(t-t_0)|<\epsilon$. We conclude that $f'(t_0)$ exists and is 0.

Using (*) again one checks that f' is continuous on R.

Letting f' play the role of f in the above argument, we see that f'' exists as a continuous function and is 0 in $E\backslash S$. Continuing in this manner we see that f is in $C^\infty(R)$.

For t in a component interval of length l of the complement of E we have

$$|f^{(n)}(t)| \leqq 2h(l)l^{-n}(4\lambda)^n N_n \leqq 2(4\lambda)^n M_n.$$

Hence $|f^{(n)}(t)| \leqq 2(4\lambda)^n M_n$ on the dense set D and thus f is in $C\{M_n\}$.

As an example of an application of our theorem we give the following:

COROLLARY. *Let E and E' be closed sets of real numbers. Then there is a continuous solution $u(x, t)$ to the heat equation, $u_{xx}=u_t$, in the $(x-t)$-plane satisfying $\{t:u(0,t)=0\}=E$ and $\{t:u_x(0,t)=0\}=E'$.*

PROOF. We define

$$u(x, t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(t)x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{g^{(n)}(t)x^{2n+1}}{(2n+1)!}$$

where f and g are in $C\{\Gamma(3n/2)\}$ with appropriate zero sets.

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