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AN OPERATOR VALUED FUNCTION SPACE INTEGRAL: A SEQUEL TO CAMERON AND STORVICK'S PAPER

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ABSTRACT. Recently Cameron and Storvick introduced and studied an operator valued function space integral related to the Feynman integral. The main theorems of their study establish the existence of the function space integral as a weak operator limit of operators defined at the first stage by finite-dimensional integrals. This paper provides a substantial strengthening of their existence theorem giving the function space integrals as strong operator limits rather than as weak operator limits.

1. **Introduction.** The function space integral referred to in the title was first defined and studied in [1]. The study of this integral has been continued in [2], [5], and [7]. We assume familiarity with [1] and adopt the notation and terminology of that paper. The basic definitions may also be found in [5]. In connection with Theorems 4 through 7 of [1], we are able to replace the weak operator limits in the conclusion with strong operator limits. The major difficulty occurs when the parameter λ is pure imaginary. Here the argument we give is an adaptation of an argument of Nelson [6, p. 335–336]; the crucial step is establishing a Poisson representation for the operators $I_\lambda(F)$ $(\operatorname{Re}\lambda>0)$ in terms of the boundary values $J_q(F)$.

One effect of the present paper is to show that the Cameron-Storvick approach [1] to the Feynman integral and the Nelson approach [6] are somewhat more closely related than they previously appeared to be; now in both approaches, strong operator limits can be used at each stage in obtaining the Feynman integral.

Hopefully, having the $J_q(F)$ as strong operator limits rather than as weak operator limits as in [1] will facilitate the study of the properties of $J_q(F)$ and its relationship with F. The finite-dimensional integrals that appear in the first stage of the definition of $J_q(F)$ are relatively concrete analytic objects; they define operators on $L_2(-\infty, \infty)$ which are the composition of a finite number of multiplication and convolution operators. The results of this paper strengthen the tie between $J_q(F)$ and these relatively simple operators.

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2. **The existence theorems.** We will first establish the strong convergence for positive λ.

THEOREM 1. *Let $\theta(t, u)$ be continuous almost everywhere on $R \equiv [a, b] \times (-\infty, \infty)$; let $\operatorname{Re} \theta(t, u) \leqq B$ on R; and let $\theta(t, u)$ be bounded on every compact subset of R. Let $F(x) = \exp[\int_a^b \theta(s, x(s))ds]$. Then for $\lambda > 0$, $I_\lambda^\sigma(F) \to I_\lambda(F)$ in the strong operator topology as* norm $\sigma \to 0$.

PROOF. Let $\lambda > 0$ and $\psi \in L_2 = L_2(-\infty, \infty)$ be given. It suffices to show that $I_\lambda^{\sigma_m}(F)\psi \to I_\lambda(F)\psi$ in L_2 norm for any sequence $\{\sigma_m\}$ of partitions such that norm $\sigma_m \to 0$. To get this result, it suffices to show that $I_\lambda^{\sigma_m}(F)\psi \to I_\lambda(F)\psi$ almost everywhere and that the sequence $\{I_\lambda^{\sigma_m}(F)\psi\}$ of L_2 functions is dominated by an L_2 function g. The first of these facts is established in [1, p. 526]. Next note that for any partition $\sigma: a = t_0 < t_1 < \cdots < t_n = b$, we have

$$
\begin{aligned}
&|(I_\lambda^\sigma(F)\psi)(\xi)| \\
&= \Big| \lambda^{n/2}[(2\pi)^n(t_1 - a) \cdots (t_n - t_{n-1})]^{-1/2} \int_{-\infty}^{\infty} \cdot (n) \cdot \int_{-\infty}^{\infty} \psi(v_n) \\
&\quad \cdot \exp\left\{ \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \theta(t, v_{j-1})dt - \lambda(v_j - v_{j-1})^2/2(t_j - t_{j-1}) \right\} dv_1 \cdots dv_n \Big| \\
&\leqq e^{B(b-a)} \lambda^{n/2}[(2\pi)^n(t_1 - a) \cdots (t_n - t_{n-1})]^{-1/2} \\
&\quad \cdot \int_{-\infty}^{\infty} \cdot (n) \cdot \int_{-\infty}^{\infty} |\psi(v_n)| \exp\left\{ -\sum_{j=1}^{n} \lambda(v_j - v_{j-1})^2/2(t_j - t_{j-1}) \right\} \\
&\qquad\qquad dv_1 \cdots dv_n \\
&= e^{B(b-a)}[\lambda/2\pi(b - a)]^{1/2} \int_{-\infty}^{\infty} |\psi(v_n)| \exp[-\lambda(v_n - \xi)^2/2(b - a)]dv_n \\
&\equiv g(\xi)
\end{aligned}
$$

where $v_0 \equiv \xi$. But $g \in L_2$ by Lemma 1 of [1]. Q.E.D.

For $\lambda \in C^+ \equiv \{\lambda : \operatorname{Re} \lambda > 0\}$, we will follow the notation of [2] and [5] and let $I_\lambda(F)$ denote the common value of $I_\lambda^{an}(F)$ and $I_\lambda^{\text{seq}}(F)$ when both exist. We are now ready to give the strengthened version of Theorem 4 of [1]. The existence of $I_\lambda(F)$ below is insured by that theorem.

THEOREM 2. *Let θ and F be as in Theorem 1 above and let $\lambda \in C^+$. Then $I_\lambda^\sigma(F) \to I_\lambda(F)$ in the strong operator topology as* norm $\sigma \to 0$.

PROOF. Let $\psi \in L_2$. Again it suffices to consider a sequence $\{\sigma_m\}$ of partitions such that norm $\sigma_m \to 0$. Note the following:

(a) $I_\lambda^{\sigma_m}(F)\psi$ is holomorphic for $\lambda\in C^+$ [1, p. 533].
(b) $\|I_\lambda^{\sigma_m}(F)\psi\|\leqq\|\psi\|e^{B(b-a)}$ [1, Lemma 6].
(c) For $\lambda>0$, $\|I_\lambda^{\sigma_m}(F)\psi-I_\lambda(F)\psi\|\to 0$.

Our result now follows from the Vitali Theorem for vector-valued functions [4, Theorem 3.14.1]. Q.E.D.

Next we give the main theorem of this paper: the strengthened version of Theorem 5 of [1].

THEOREM 3. *Again let θ and F be as in Theorem 1 above. Then there exists a null subset A of $(-\infty, \infty)$ containing 0 and such that for all $q\notin A$, $J_q(F)$ is the strong operator limit of $I_\lambda(F)$ as $\lambda\to -iq$ along a horizontal line in C^+; that is, $\lim_{p\to 0^+}\|I_{p-iq}(F)\psi-J_q(F)\psi\|=0$ for every $q\notin A$ and every $\psi\in L_2$.*

PROOF. Let $\psi\in L_2$. By Theorem 5 of [1], for almost all $q\in(-\infty, \infty)$, $I_{p-iq}(F)\psi\to J_q(F)\psi$ weakly as $p\to 0^+$. Further $\|J_q(F)\psi\|\leqq\|\psi\|e^{B(b-a)}$. Now for $\lambda\in C^+$, $I_\lambda(F)\psi$ is weakly measurable in λ [4, Definition 3.5.4] since $I_\lambda(F)\psi$ is analytic in λ. Hence $J_q(F)\psi$ is a weakly measurable function of q. Since L_2 is separable, it follows that $J_q(F)\psi$ is strongly measurable [4, Corollary 2, p. 73]. Then it also follows that $\|J_q(F)\psi\|$ is a measurable function of q [4, Theorem 3.5.2]. Also for each $\lambda=x-yi\in C^+$, $g(q)\equiv(\pi^{-1}x/[x^2+(y-q)^2])J_q(F)\psi$ is Bochner integrable since by [4, Theorem 3.7.4]

$$\int_{-\infty}^{\infty}\|g(q)\|dq\leqq\|\psi\|e^{B(b-a)}\int_{-\infty}^{\infty}\pi^{-1}x/[x^2+(y-q)^2]dq=\|\psi\|e^{B(b-a)}.$$

Next we wish to establish the Poisson formula

$$I_{x-yi}(F)\psi=\int_{-\infty}^{\infty}(\pi^{-1}x/[x^2+(y-q)^2])J_q(F)\psi dq. \tag{1}$$

Since both sides of (1) are in L_2, it suffices to show that we get equality when we take the inner product of the two sides with an arbitrary $\phi\in L_2$. But by [4, Theorem 3.7.12 and following comment] we have

$$\left(\int_{-\infty}^{\infty}g(q)dq, \phi\right)=\int_{-\infty}^{\infty}(\pi^{-1}x/[x^2+(y-q)^2])(J_q(F)\psi, \phi)dq.$$

Finally this last expression equals $(I_{x-yi}(F)\psi, \phi)$ by the classical scaler-valued Poisson representation [3, p. 455]. Using (1) and [4, Theorem 3.7.6] we have for every $\lambda=x-yi\in C^+$,

$$(2)\qquad \|I_{x-yi}(F)\psi\| \leqq \int_{-\infty}^{\infty} (\pi^{-1}x/[x^2+(y-q)^2])\|J_q(F)\psi\|dq.$$

Now using the fact that $\|J_q(F)\psi\|$ is a real-valued bounded, measurable function of q and [3, Lemma 19.2.1], we have for almost all $y \in (-\infty, \infty)$,

$$(3)\qquad \lim_{x\to 0^+} \int_{-\infty}^{\infty} (\pi^{-1}x/[x^2+(y-q)^2])\|J_q(F)\psi\|dq = \|J_y(F)\psi\|.$$

Thus by (2) and (3), we have for almost all $y \in (-\infty, \infty)$,

$$(4)\qquad \limsup_{x\to 0^+} \|I_{x-yi}(F)\psi\| \leqq \|J_y(F)\psi\|.$$

By enlarging the null set if necessary we can also insure that $I_{x-yi}(F)\psi \to J_y(F)\psi$ weakly as $x \to 0^+$. But then $\|J_y(F)\psi\| \leqq \liminf_{x\to 0^+} \|I_{x-yi}(F)\psi\|$. Putting this together with (4), we have

$$\lim_{x\to 0^+} \|I_{x-yi}(F)\psi\| = \|J_y(F)\psi\|.$$

This last equality and the weak convergence imply the convergence in norm of $I_{x-yi}(F)\psi$ to $J_y(F)\psi$.

The null set in the construction above may well depend upon ψ, and we would like to eliminate this feature. Let $\{\psi_n\}$ be a countable dense subset of L_2 and obtain a null set A_n as above for each n. Let A be the union of the A_n's. Using the fact that the operators $I_\lambda(F)$ and $J_q(F)$ are bounded by $e^{B(b-a)}$, one may show without much difficulty that A has the properties referred to in the conclusion of the theorem. Q.E.D.

We finish by giving the strengthened forms of Theorems 6 and 7 of [1]. We again have strong operator limits in the conclusion rather than weak operator limits. The proof will not be included as it follows almost word for word the proofs above except that the bound $e^{B(b-a)}$ on the operators is replaced by $\sum_{n=0}^{\infty} |a_n| [M(b-a)]^n$.

THEOREM 4. *Let $\theta(t, u)$ be continuous almost everywhere on R and let $|\theta(t, u)| \leqq M$ on R; let $G(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function which is analytic in a disk centered at the origin of radius greater then $M(b-a)$. Let $F(x) \equiv G(\int_a^b \theta(s, x(s))ds)$. Then the conclusions of Theorems* 1, 2 *and* 3 *above hold.*

BIBLIOGRAPHY

1. R. H. Cameron and D. A. Storvick, *An operator value function space integral and a related integral equation*, J. Math. Mech. **18** (1968), 517–552. MR **38** #4643.

2. ———, *An integral equation related to the Schroedinger equation with an application to integration in function space*, Bochner Memorial Volume (to appear).

3. E. Hille, *Analytic function theory*. Vol. II, Introduction to Higher Math., Ginn, Boston, Mass., 1962. MR **34** #1490.

4. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, rev. ed., Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R.I., 1957. MR **19**, 664.

5. G. W. Johnson and D. L. Skoug, *Operator-valued Feynman integrals of certain finite-dimensional functionals*. Proc. Amer. Math. Soc. **24** (1970), 774–780.

6. E. Nelson, *Feynman integrals and the Schrödinger equation*, J. Mathematical Phys. **5** (1964), 332–343. MR **28** #4397.

7. D. L. Skoug and G. W. Johnson, *Operator-valued Feynman integrals of finite-dimensional functionals*, Pacific J. Math. **34** (1970), 415–426.

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