PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 3, March 1971

A COMPARISON LEMMA FOR HIGHER ORDER TRAJECTORY DERIVATIVES[1]

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ABSTRACT. A basic result from higher order differential inequalities is used to obtain a comparison lemma, useful when higher order trajectory derivatives of Liapunov functions are known.

1. **Introduction.** Several recent results have focused interest on the question of obtaining stability criteria in terms of higher ordered derivatives of Liapunov functions. Yorke [1] and Kudaev [2] have investigated conditions on the second trajectory derivative and Butz [3] discussed a condition involving the third derivative. In addition, Butz referred to the availability of derivative information from on-line computation of v-functions as motivation for further study of conditions of this type. In the following a basic result from the theory of higher ordered differential inequalities is used to obtain a comparison type lemma involving trajectory derivatives of arbitrary order. It is shown that application of the lemma offers several potential advantages in the study of solution behavior by means of v-functions.

2. **Notation and definitions.** Consider the system of first order differential equations

$$\dot{x} = f(t, x) \tag{1}$$

and the mth order comparison equation

$$u^{(m)}(t) = w(t, u, u', \cdots, u^{(m-1)}) \tag{2}$$

where x, f belong to R^n and t, u are scalar. Assume f continuous on $D_r = \{(t, x) | 0 \leqq t \leqq T < +\infty, |x| < r\}$ and the right side of (2) continuous on $[0, T] \times R^m$. A solution of (1) satisfying the initial condition x_0 at t_0 will be denoted by $x(t, t_0, x_0)$ and a solution of (2) satisfying the initial conditions $u^{(j)}(t_0) = u_j$ $(j = 0, 1, 2, \cdots, m-1)$ will be denoted by $u(t, t_0, U_0)$.

Received by the editors April 22, 1970.

AMS 1969 *subject classifications.* Primary 3440; Secondary 3490.

Key words and phrases. High order trajectory derivatives, vector v-functions, comparison lemma, stability properties.

[1] This work was supported by the National Aeronautics and Space Administration under Grant No. NGR-45-002-016.

The following are adaptations of definitions from first order differential inequalities to inequalities of higher order (cf. [4]).

DEFINITION 1. The scalar function $g(x)$, $x\in R^n$, will be said to be of type W^* on a set $S\subset R^n$ if $g(a)\leqq g(b)$ for any a, b in S such that $a_n=b_n$, $a_i\leqq b_i$ $(i=1, 2, \cdots, n-1)$.

DEFINITION 2. A solution $u_m(t, t_0, U_0)$ is called a right maximal solution of (2) on an interval $[t_0, \alpha)$ if

$$\overset{(j)}{u}(t) \leqq \overset{(j)}{u_m}(t, t_0, U_0), \qquad t\in [t_0, \alpha)\cap [t_0, \alpha^*),$$

for any solution $u(t)$ satisfying the initial conditions $u^{(j)}(t_0)\leqq u_j$ $(j=0, 1, 2, \cdots, m-1)$ and defined on $[t_0, \alpha^*)$.

DEFINITION 3. The scalar function $a(r)$, defined for $0\leqq r$, will be said to be of type K if it is continuous and strictly increasing for $r\geqq 0$ and if $a(0)=0$ while $a(r)\rightarrow\infty$ as $r\rightarrow\infty$.

In the following, the vector inequality $a\geqq b$ for a, $b\in R^n$ implies $a_i\geqq b_i$ for all $i=1, 2, \cdots, n$.

3. A comparison lemma.

The following is given relative to (1) and (2):

LEMMA. *Let $v: D_r\rightarrow R$ and let $v\in C^m$, $f\in C^{m-1}$ on D_r. Let w of* (2) *be of type W^* in S for each t, where $S=\{(t, v(t, x), v'(t, x), \cdots, v^{(m-1)}(t, x)) \mid (t, x)\in D_r\}$ and*

$$v^{(j)}(t, x) = \frac{\partial v^{(j-1)}}{\partial t} + \frac{\partial v^{(j-1)}}{\partial x}\cdot f(t, x). \tag{3}$$

Suppose

$$v^{(m)}(t, x) \leqq w(t, v, v', \cdots, v^{(m-1)}) \tag{4}$$

for $(t, x)\in D_r$ and set $v^{(j)}(0, x_0)=u_j$. Let J denote the maximal interval of existence of the right maximal solution $u_m(t, 0, U_0)$. Then

$$\overset{(j)}{v}(t, x(t, 0, x_0)) \leqq \overset{(j)}{u_m}(t, 0, U_0) \tag{5}$$

for each $t\in J\cap[0, T]$.

The proof of the lemma follows almost immediately upon application of a basic theorem for nth order differential inequalities, due mainly to Kamke (cf. [6, pp. 60–61]), and a standard proof showing the equivalence of the derivative (3) to the kth trajectory derivative $v^{(k)}(t, x(t, 0, x_0))$, [5, p. 3]. It is possible to weaken the conditions on v and f somewhat by use of Yoshizawa's trajectory derivative, however the derivative (3) is more likely to be of use in the applications.

REMARK 1. The above lemma remains true if the inequalities "$\leqq$" are replaced throughout by "$\geqq$" and if "right maximal solution" is replaced by "right minimal solution."

Since the inequality (4) implies the system of first order inequalities

$$\begin{aligned} \dot{v}_1 &= v_2, \\ \dot{v}_2 &= v_3, \\ &\vdots \\ \dot{v}_m &\leqq w(t, v_1, v_2, \cdots, v_m) \end{aligned}$$

it is natural to view higher order derivatives as a means of obtaining vector v-functions. That is, assuming (5) and appropriate additional assumptions on the functions v_i, it is possible to invoke numerous theorems (e.g. [6, pp. 267–311]) to show that the existence of a certain solution property for (2) (e.g. exponential stability) implies the corresponding property for (1). There are several practical difficulties, however, which limit the value of this approach. One such difficulty is created by having to require the right side of (4) to be of type W^*. This restriction essentially eliminates the use of linear comparison equations for stability investigations, since the characteristic polynomial of such an nth order comparison equation must be of the form

$$s^n \pm a_{n-1}s^{n-1} - a_{n-2}s^{n-2} - \cdots - a_1 s - a_0 = 0 \qquad (a_i \geqq 0)$$

while a necessary condition for stability is that all coefficients be positive. Another difficulty is encountered in satisfying the additional properties, such as definiteness properties, required of the functions $v^{(j)}$ in (5). In fact, these assumptions will in general be either equivalent or more difficult to satisfy, than those of the classical theorems of the direct method.

By use of the lemma it is possible to obtain results, such as the following, more suited to the application of higher ordered trajectory derivative information.

THEOREM 1. *Let v, f, w satisfy the conditions of the lemma and suppose $f(t, 0) \equiv 0$ on $[0, \infty)$. In addition, suppose*

(i) $\alpha_1(|x|) \leqq v(t, x) \leqq \alpha_2(|x|)$, $(t, x) \in D_r$, $\alpha_1, \alpha_2 \in K$,

(ii) *that solutions of* (2) *satisfy the inequality* $0 < u(t, 0, V_0) < \alpha_3(v_0)$, $v_0 = v(0, x_0)$, $\alpha_3 \in K$, *where* $V_0 = \{v(0, x_0), v'(0, x_0), \cdots, v^{(m-1)}(0, x_0)\}$ *and* $|x_0| < r_1$, $0 < r_1 < r$. *Then*

(iii) *the zero solution of* (1) *is stable at* $t = 0$.

PROOF. The proof follows by defining δ to be the composition function $\delta=\alpha_2^{-1}\cdot\alpha_3^{-1}\cdot\alpha_1$ so that $|x_0|<\delta$ implies $v(0, x_0)=v_0\leqq\alpha_2(|x_0|)<(\alpha_3^{-1}\cdot\alpha_1)(\epsilon)$. Then $u(t, 0, V_0)<\alpha_3(v_0)<\alpha_1(\epsilon)$ and, by the lemma,

$$\alpha_1(|x(t, 0, x_0)|) \leqq v(t, x(t, 0, x_0)) \leqq u(t, 0, V_0) < \alpha_1(\epsilon),$$

i.e., $|x(t, 0, x_0)|<\epsilon$ for $t\in[0, \infty)$.

In the same manner, it is possible to obtain similar results for most of the remaining stability properties. However, before proceeding further it is useful to examine the conditions of the theorem relative to the applications.

REMARK 2. Condition (i) is certainly more realistic than the corresponding conditions resulting from direct application of vector v-function theorems, where inequalities of the same type must be satisfied for each of the derivatives of v. Condition (ii) offers some improvement also, in that the solution property required of the comparison system (2) need hold only on the set V_0 of initial values. At least in theory then, it should be possible to satisfy condition (ii) even though the comparison system (2) may not be stable at $t=0$. In practice a convenient means of verifying the condition would be through showing the solution property to hold on some neighborhood of the origin in R^m, which contains V_0 for r_1 sufficiently small. Unfortunately, this again leads to difficulties with linear comparison systems, since condition (ii) would then require the real part of every root of the characteristic equation to be nonpositive.

REMARK 3. From the proof to the theorem it can be seen that once the inequality (4) is established only the initial values $v^{(j)}(0, x_0)$ are required to obtain an estimate for $v(t, x(t, 0, x_0))$ for $t>0$. Since the initial values $v^{(j)}(0, x_0)$ are easily calculated, the lemma might profitably be used to obtain estimates on the behavior of particular solutions $x(t, 0, x_0)$. While such estimates might not imply stability in the sense of any of the formal definitions, such information may well be of equal or greater importance to a given application. The following theorem should prove useful.

THEOREM 2. *Let* $v(t, x)=x'Hx$ *where* H *is positive definite and suppose the trajectory derivatives of* v, *formed relative to* (1), *satisfy*

$$v^{(m)} \pm a_{m-1}v^{(m-1)} - a_{m-2}v^{(m-2)} - \cdots - a_0 v \leqq 0 \tag{6}$$

for $(t, x)\in D_r$ *and* $a_i\geqq 0$ $(i=0, 1, 2, \cdots, m-1)$. *Then*

$$|x(t, 0, x_0)|^2 \leqq \frac{1}{\gamma}\sum_{k=1}^{m} c_k u_k(t) \tag{7}$$

for all $t \geqq 0$, $(t, x(t, 0, x_0)) \in D_r$, *where* $\sum_{k=1}^{m} c_k u_k(t)$ *is a particular solution of the comparison equation*

$$u^{(m)} \pm a_{m-1}u^{(m-1)} - a_{m-2}u^{(m-2)} - \cdots - a_0 u = 0, \tag{8}$$

γ *is the minimal eigenvalue of* H *and where the constants* c_k *satisfy the system of inequalities*

$$v^{(j)}(0, x_0) \leqq \sum_{k=1}^{m} c_k u_k^{(j)} \qquad (j = 0, 1, \cdots, m-1). \tag{9}$$

The proof follows immediately from the lemma, since the comparison equation $u^{(m)} = a_0 u + a_1 u' + \cdots \pm u^{(m-1)}$ has right side of type W^* and since the positive definite quadratic form $v(t, x) = x'Hx$ will satisfy an estimate of the form $\gamma x'x \leqq v(t, x)$. Note that the characteristic equation of (8) will always possess at least one root with positive real part, so that in certain applications it may be desirable to set the corresponding coefficient $c_k = 0$ in (7) and (9). Assuming solutions exist to (9) for $(0, x_0) \in D_r$, the estimate (7) will hold. Note also that since v is positive definite, the condition (6) can always be satisfied assuming $v^{(m)}(t, x)$ bounded on D_r.

REMARK 4. Theorem 2 remains valid if the inequalities "$\leqq$" are replaced throughout by "$\geqq$".

EXAMPLE. Consider the (unstable) system of two first order differential equations

$$\dot{x}_1 = -3x_1 + 4x_2, \qquad \dot{x}_2 = -2x_1 + 3x_2$$

and let $v = x_1^2 + x_2^2$. Then

$$\dot{v} = -6x_1^2 + 4x_1x_2 + 6x_2^2$$

and

$$\ddot{v} = 28x_1^2 - 72x_1x_2 + 52x_2^2.$$

Since $\ddot{v}$ is positive semidefinite, by Theorem 2 and Remark 4,

$$|x(t, 0, x_0)| \geqq |x_0|$$

for every $(0, x_0)$ such that $\dot{v}(0, x_0) \geqq 0$.

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