A PRODUCT INTEGRAL REPRESENTATION
FOR AN EVOLUTION SYSTEM

J. V. HEROD

Abstract. This paper provides a product integral representation for a nonlinear evolution system. The representation is valid for expansive systems and provides an analysis in the nonexpansive case which is different from ones previously discovered.

In [7], D. Rutledge obtains a product integral representation for a nonexpansive, nonlinear semigroup. In [6], Neuberger gets such a representation for expansive semigroups by first considering nonexpansive evolution systems. This paper obtains a product integral representation for an expansive evolution system $M$. In this development, it is not required that $\lim_{h \to 0} h^{-1}[M(h, 0) - 1]P$ exist. As a corollary to Theorem 3, a statement equivalent to the statement that $M$ is nonexpansive is found.

Suppose that $\{G, +, | \cdot |\}$ is a complete, normed, Abelian group and that $S$ is the set of real numbers. If $f$ is a function from $S$ to $G$ and $a > b$, then denote the range of the restriction of $f$ to $[b, a]$ by $f([b, a])$. Also, the statement that $\{s_p\}_0^n$ is a subdivision of $\{a, b\}$ means that $s$ is a decreasing sequence with $s(0) = a$ and $s(n) = b$. The statement that $t$ is a refinement of the subdivision $s$ means that $t$ is a subdivision of $\{a, b\}$ and that there is an increasing sequence $u$ so that $s(p) = t(u(p))$ for $1 \leq p \leq n$. Finally, if $\{f_p\}_1^n$ is a sequence of functions from $G$ to $G$ and $g$ is in $G$, then

$$\left( \prod_{p=1}^n f_p \right)(g) = f_1(f_2(\cdots f_n(g))).$$

An evolution system on $G$ is a function $M$ with domain contained in $S \times S$ so that if $x \geq y$ then $M(x, y)$ is a function from $G$ to $G$ having the following properties:

1) if $x \geq y \geq z$ then $M(x, y)M(y, z) = M(x, z)$ and $M(x, x) = 1$, the identity function on $G$, and

2) if $t$ is a number and $P$ is in $G$ then the function $g$ given by $g(x) = M(x, t)P$, for all $x \geq t$, is continuous.

Presented to the Society, January 23, 1970 under the title A product integral representation for an expansive evolution system; received by the editors April 21, 1970.

AMS 1969 subject classifications. Primary 4750; Secondary 3495.

Keywords and phrases. Product integral, evolution system, semigroups of operators.

Copyright © 1971. American Mathematical Society
In order to obtain a product integral representation for the evolution system $M$, two additional conditions are used:

(3) there is an increasing, continuous function $\beta$ and a subset $D$ of $G$ so that

(a) if $P$ is in $D$ and $x>y$ then $M(x, y)P$ is in $D$, and

(b) if $P$ is in $D$, $e>0$, $a>b$, and $Q$ is in $M([b, a], b)P$ then there is a positive number $\delta$ so that if $R$ is in $M([b, a], b)P$, $|Q-R|<\delta$, and $a \geq x \geq y \geq b$, then

$$ | [M(x, y) - 1]R - [M(x, y) - 1]Q | \leq [\exp(\beta(x) - \beta(y)) - 1] \cdot e, $$

and

(4) there is a nondecreasing, continuous function $\alpha$ so that if $x>y$ and $\exp(\alpha(x) - \alpha(y))<2$, then $2-M(x, y)$ has range all of $G$ and, if $P$ and $Q$ are in $G$, then

$$ | [2 - \exp(\alpha(x) - \alpha(y))] \cdot | P - Q | \leq | [2 - M(x, y)]P - [2 - M(x, y)]Q |. $$

**Remark.** It follows from condition (4) that if $\exp(\alpha(x) - \alpha(y))<2$, then $[2-M(x, y)]^{-1}$ has domain all of $G$, and if $P$ and $Q$ are in $G$ then

$$ | [2 - M(x, y)]^{-1}P - [2 - M(x, y)]^{-1}Q | \leq | 2 - \exp(\alpha(x) - \alpha(y)) |^{-1} | P - Q |. $$

In this paper, the following three theorems are proved.

**Theorem 1.** Suppose that $P$ is in $D$, $a>b$, and $M$ satisfies conditions (1)-(4). It follows that $M(a, b)P=X$ in the sense that if $e>0$, then there is a subdivision $s$ of $\{a, b\}$ so that if $\{t^*_l\}$ is a refinement of $s$ then

$$ M(a, b)P - \prod_{p=1}^n [2 - M(t_{p-1}, t_p)]^{-1}P | < e. $$

**Theorem 2.** Suppose that $M$ satisfies conditions (1)-(4), if $x>y$ then $M(x, y)$ is continuous from $G$ to $G$, $D$ is dense in $G$, $a>b$, and $P$ is in $G$, it follows that $M(a, b)P=\prod_{p=1}^n [2 - M]^{-1}P$.

**Theorem 3.** Suppose that $G$ is a Banach space, $M$ satisfies conditions (1)-(3). If $x>y$ then $M(x, y)$ is continuous from $G$ to $G$, $D$ is dense in $G$, and $\rho$ is a continuous, real valued function which is of bounded variation on each interval. These are equivalent:

(a) If $x>y$ and $P$ and $Q$ are in $G$ then

$$ | M(x, y)P - M(x, y)Q | \leq \exp(\rho(x) - \rho(y)) \cdot | P - Q |. $$
(b) If $x > y$ and $\exp(\rho(x) - \rho(y)) < 2$, then $2 - M(x, y)$ has range all of $G$ and, if $P$ and $Q$ are in $G$, then

$$
[2 - \exp(\rho(x) - \rho(y))] \cdot |P - Q| \leq |2 - M(x, y)|P - |2 - M(x, y)|Q| .
$$

**Indication of Proofs.** The following inequality is important in what follows; it may be established after considering the polynomial $P(z) = 1 - 2z^2 + z^3$. It is labeled Lemma 1 for later reference.

**Lemma 1.** If $x$ is a number and $1 \leq x \leq (1 + \sqrt{5})/2$ then $[2 - x]^{-1} \leq x^2$.

In the definitions and lemmas which follow, suppose that $M$ satisfies conditions (1)-(4), $a > b$, and $\epsilon > 0$.

**Definition.** Define functions $\delta$ and $B$ as follows: if $P$ is in $D$ and $a \leq z \leq b$ then $\delta(z, P)$ is the largest number $d$ not exceeding 1 so that if $Q$ is in $M([z, a], z)$, $P\setminus Q - R < d$, and $a \leq x \leq y \leq z$ then

$$
|M(x, y) - 1|Q - |M(x, y) - 1|P : \leq \exp(\beta(x) - \beta(y)) - 1\cdot \epsilon .
$$

Also, $B(z, P)$ is the largest number $u$ not exceeding $a$ so that if $u > v > z$ then $|M(v, z)P - P| < 5(z, P)$.

**Remark.** Note that the existence of $\delta$ follows from condition (3) and of $B$ follows from condition (2).

**Lemma 2.** Suppose that $P$ is in $D$. If $a \leq x \leq b$, $\{t_p\}^n_0$ is a subdivision of $\{B(x, P), x\}$, and $j$ is an integer in $[1, n]$, then

$$
| M(t_{j-1}, t_j) - 1 | M(t_j, t_n) P - | M(t_{j-1}, t_j) - 1 | P | \leq \exp(\beta(t_{j-1}) - \beta(t_j)) - 1\cdot \epsilon .
$$

**Indication of Proof.** If $\{t_p\}^n_0$ is a subdivision of $\{B(x, P), x\}$ and $j$ is an integer in $[1, n]$ then $x \leq t_j < B(x, P)$. Thus $|M(t_j, x)P - P| < 5(x, P)$. Now, $M(t_j, x)P$ is in $M([x, a], x)P$, so if $a \leq u \leq v \leq x$ then

$$
| M(u, v) - 1 | M(t_j, x) P - | M(u, v) - 1 | P | \leq \exp(\beta(u) - \beta(v)) - 1\cdot \epsilon .
$$

**Lemma 3.** Suppose that $P$ is in $D$, $\{t_p\}^n_0$ is an increasing sequence with values in $[b, a]$ and limit $z$. There is a positive integer $N$ so that if $n > N$ then $B(t_n, M(t_n, b)P) \geq z$.

**Indication of Proof.** Suppose that $P$ is in $D$ and $t$ is an infinite increasing sequence with values in $[b, a]$ and limit $z$. The fact that $\{M(t_p, b)P\}^\infty_{p=0}$ converges in $G$ and has limit $M(z, b)P$ follows from
condition (2). Let $Q$ be $M(z, b)P$. Since $Q$ is in $M([b, a], b)P$, there is a number $d$ so that $0 < d < 1$ and, if $|R - Q| < d$ and $R$ is in $M([b, a], b)P$ and $a \geq x \geq y \geq b$, then

$$| [M(x, y) - 1]Q - [M(x, y) - 1]R | \leq [\exp(\beta(x) - \beta(y)) - 1] \cdot \epsilon/2.$$ 

Let $w$ be so that if $z \geq u \geq w$ then $|Q - M(u, b)P| < d/4$. Let $n$ be so that $t_n > w$. First, $\delta(t_n, M(t_n, b)P) \geq d/2$ because: suppose $R$ is in $M([t_n, a], b)P$ and $|R - M(t_n, b)P| < d/2$. Then $|R - Q| < d$ so that if $a \geq x \geq y \geq b$ then

$$| [M(x, y) - 1]M(t_n, b)P - [M(x, y) - 1]R | \leq [\exp(\beta(x) - \beta(y)) - 1] \cdot \epsilon/2 + \epsilon/2.$$ 

Finally, $B(t_n, M(t_n, b)P) \geq z$ because: suppose that $t_n \leq v \leq z$. Then

$$| M(v, t_n)M(t_n, b)P - M(t_n, b)P | \leq | M(v, t_n)M(t_n, b)P - Q | + | Q - M(t_n, b)P | \leq d/4 + d/4 \leq \delta(t_n, M(t_n, b)P).$$

**Lemma 4.** Suppose that $P$ is in $D$. There is a subdivision $u$ of $\{a, b\}$ so that if $\{t_p\}$ is a refinement of $u$ and $p$ is an integer in $[1, n]$ then

$$| [M(t_{p-1}, t_p) - 1]M(t_{p-1}, b)P - [M(t_{p-1}, t_p) - 1]M(t_p, b)P | \leq [\exp(\beta(t_{p-1}) - \beta(t_p)) - 1] \cdot 2\epsilon.$$ 

**Indication of Proof.** Suppose that $P$ is in $D$. By the previous lemma, there is a subdivision $\{u_q\}$ of $\{a, b\}$ so that if $q$ is an integer in $[1, m]$ then $u_{q-1} = B(u_q, M(u_q, b)P)$. Let $\{t_p\}$ be a refinement of $u$ and $p$ be an integer in $[1, n]$. Let $q$ be an integer in $[1, m]$ so that $u_{q-1} \geq t_{p-1} > t_p \geq u_q$. Then $| M(t_{p-1}, b)P - M(u_q, b)P | < \delta(u_q, M(u_q, b)P)$ and $| M(t_p, b)P - M(u_q, b)P | < \delta(u_q, M(u_q, b)P)$. Hence, if $a \geq x \geq y \geq u_q$, then

$$| [M(x, y) - 1]M(t_{p-1}, b)P - [M(x, y) - 1]M(t_p, b)P | \leq [\exp(\beta(x) - \beta(y)) - 1] \cdot 2\epsilon.$$ 

**Indication of Proof of Theorem 1.** Suppose that $P$ is in $D$. Let $u$ be a subdivision of $\{a, b\}$ as indicated in Lemma 4, $\{s_p\}$ be a refinement of $u$ so that if $p$ is an integer in $[1, m]$ then $\exp(\alpha(s_{p-1}) - \alpha(s_p)) < (1 + \sqrt{5})/2$, and $\{t_p\}$ be a refinement of $s$. By Lemma 1, if $p$ is an integer in $[1, n]$ and $P$ and $Q$ are in $G$, then

$$| [2 - M(t_{p-1}, t_p)]^{-1}P - [2 - M(t_{p-1}, t_p)]^{-1}Q | \leq \exp(2[\alpha(t_{p-1}) - \alpha(t_p)]) \cdot | P - Q |.$$
\[
\prod_{p=1}^{n} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} P - M(a, b) P = \left| \sum_{j=1}^{n} \left\{ \prod_{p=1}^{j} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} M(t_{n+1-j}, b) P \right. \right.
\]

\[
- \left. \left. \prod_{p=1}^{n-j} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} M(t_{n-j}, b) P \right\} \right| \]

\[
\leq \sum_{j=1}^{n} \exp(2[\alpha(a) - \alpha(t_{n+1-j})]) \cdot \left| M(t_{n+1-j}, b) P - \left[ 2 - M(t_{n-j}, t_{n+1-j}) \right] M(t_{n-j}, b) P \right|
\]

\[
= \sum_{j=1}^{n} \exp(2[\alpha(a) - \alpha(t_{n+1-j})]) \cdot \left[ M(t_{n-j}, t_{n+1-j}) - 1 \right] M(t_{n-j}, b) P
\]

\[
- \left[ M(t_{n-j}, t_{n+1-j}) - 1 \right] M(t_{n+1-j}, b) P \right| \]

\[
\leq \sum_{j=1}^{n} \left[ \exp(2[\alpha(a) - \alpha(t_{n+1-j})]) \cdot \left[ \exp(\beta(t_{n-j}) - \beta(t_{n+1-j}) \right) - 1 \right] \cdot 2\epsilon
\]

\[
\leq \exp(2[\alpha(a) - \alpha(b)]) \cdot \left[ \exp(\beta(a) - \beta(b) \right) - 1 \right] \cdot 2\epsilon.
\]

To see this last inequality, one should note Lemma 2.2 of [4].

\textbf{Indication of Proof of Theorem 2.} Suppose that \(P\) and \(Q\) are in \(G\), \(a > b\), and \(\{t_p\}_{p=1}^{n}\) is a subdivision of \(\{a, b\}\) so that, if \(p\) is an integer in \(\{1, n\}\), then \(2 - M(t_{p-1}, t_p)\) has domain all of \(G\).

\[
M(a, b) P \leq \prod_{p=1}^{n} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} P \left| \right| M(a, b) P - M(a, b) Q \right|
\]

\[
+ \left| \prod_{p=1}^{n} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} Q - \prod_{p=1}^{n} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} P \right|
\]

\[
+ \left| \prod_{p=1}^{n} \left[ 2 - M(t_{p-1}, t_p) \right]^{-1} Q - M(a, b) Q \right|.
\]

Thus, if \(D\) is dense in \(G\) and \(M(a, b)\) is continuous from \(G\) to \(G\), it follows from Lemma 1 that \(M(a, b) P = \prod_{a}^{b} [2 - M]^{-1} P\).

\textbf{Lemma 5.} If \(\rho\) is a continuous function from \(S\) to \(S\) and is of bounded variation on each interval of \(S\), \(a > b\), and \(\epsilon > 0\), then there is a subdivision \(s\) of \(\{a, b\}\) so that \(\{t_p\}_{p=1}^{n}\) is a refinement of \(s\) then
\[
\left| \exp(\rho(a) - \rho(b)) - \prod_{p=1}^{n} \left[ 2 - \exp(\rho(t_{p-1}) - \rho(t_{p})) \right]^{-1} \right| < \varepsilon.
\]

**Indication of Proof.** Notice that if \( \rho \) is continuous and of bounded variation on each interval of \( S \), \( a > b \), and \( \{t_p\}_0^n \) is a subdivision of \( \{a, b\} \) so that, if \( \rho \) is an integer in \([1, n]\), then \( \exp(\rho(t_{p-1}) - \rho(t_{p})) < 2 \) then

\[
\prod_{p=1}^{n} \left[ 2 - \exp(\rho(t_{p-1}) - \rho(t_{p})) \right]^{-1} \leq \prod_{p=1}^{n} \left[ 2 - \exp\left( \int_{t_p}^{t_{p-1}} |d\rho| \right) \right]^{-1}.
\]

With techniques similar to those used in the proof of Theorem 1, it can be shown that, if

\[
\exp\left( \int_{t_p}^{t_{p-1}} |d\rho| \right) < \frac{1 + \sqrt{5}}{2} \quad \text{for } \rho = 1, 2, \ldots, n,
\]

then

\[
\left| \prod_{p=1}^{n} \left[ 2 - \exp(\rho(t_{p-1}) - \rho(t_{p})) \right]^{-1} - \exp(\rho(a) - \rho(b)) \right|
\]

\[
\leq \exp\left( 3 \int_{a}^{b} |d\rho| \right) \cdot \sum_{j=1}^{n} \left| \exp(\rho(t_{n-j}) - \rho(t_{n+1-j})) - 1 \right|^2.
\]

The conclusion of the lemma follows.

**Indication of Proof of Theorem 3.** Suppose that \( G \) is a Banach space and that \( \rho \) is a function from \( S \) to \( S \) which is continuous and of bounded variation on each interval of \( S \). Suppose also that \( x > y \) and that \( M(x, y) \) is a function from \( G \) to \( G \) having the property that if \( P \) and \( Q \) are in \( G \) then \( |M(x, y)P - M(x, y)Q| \leq \exp(\rho(x) - \rho(y)) |P - Q| < 2|P - Q| \). As in Lemma 1 of [5], let \( X \) be in \( G \) and \( K(Z) \) be \(.5[X + M(x, y)Z] \) for each \( Z \) in \( G \). Then \( K \) is a contraction mapping and there is only one member \( Z \) of \( G \) so that \( 2Z - M(x, y)Z = X \). Furthermore, if \( P \) and \( Q \) are in \( G \), then

\[
|Q - P| \leq .5 \left[ 2 - M(x, y) \right] Q - [2 - M(x, y)] P
\]

\[
+ .5 \exp(\rho(x) - \rho(y)) |P - Q| .
\]

Consequently, in Theorem 3, statement (a) implies statement (b). Finally, with \( G \) and \( \rho \) as supposed above, if \( M \) satisfies conditions (1)-(3), \( D \) is dense in \( G \), statement (b) of Theorem 3 holds, and \( x > y \), then, by Theorem 2, \( M(x, y)P = \prod_{v} [2 - M]^{-1} P \) for each \( P \) in \( G \) and, by Lemma 5,
A PRODUCT INTEGRAL REPRESENTATION

\[ \prod^v_2 [2 - M]^{-1} P - \prod^v_2 [2 - M]^{-1} Q \leq \exp(\rho(x) - \rho(y)) \left| P - Q \right|. \]

This completes the proof of Theorem 3.

**Examples.**

**Example 1.** Let \( G \) be a Banach space and \( T \) be a one-parameter semigroup of nonlinear transformations on \( G \). That is, \( T \) is a function from \([0, \infty)\) to the set of continuous transformations from \( G \) to \( G \) which satisfies

1. \( T(x) T(y) = T(x+y) \) if \( x, y \geq 0 \),
2. if \( P \) is in \( G \) and \( g_p(x) = T(x) P \) for all \( x \) in \([0, \infty)\) then \( g_p \) is continuous and \( \lim_{x \to 0^+} g_p(x) = P \),
3. \( |T(x) P - T(x) Q| \leq |P - Q| \) if \( x \geq 0 \) and \( P \) and \( Q \) are in \( G \), and
4. there is a dense subset \( D \) of \( G \) such that if \( P \) is in \( D \) then \( g_p \) is continuous with domain \([0, \infty)\). By Theorem 2, if \( P \) is in \( D \) and \( x > 0 \), then \( T(x) P = \prod^0_2 [2 - T(\cdot dI)]^{-1} P \). Compare with Theorem 2 of [7].

**Example 2.** Let \( f \) be an increasing function from the real numbers onto the real numbers so that \( f' \) is continuous and nonincreasing. Suppose also that \( g \) is increasing and continuous, and that, for \( x > y \) and \( P \) a real number,

\[ M(x, y) P = f(g(x) - g(y) + t\left| P \right|). \]

\( M \) satisfies (1)-(4) but \( \lim_{h \to 0^+} h^{-1} [M(h, 0) - 1] P \) may not exist. Compare Example 2 of [8], Example 3.4 of [1], and Theorem A of [6].

**Example 3.** In case \( M \) satisfies conditions (1) and (2) and if \( P \) and \( Q \) are in \( G \) and \( x > y \), then \[ \left| [M(x, y) - 1] P - [M(x, y) - 1] Q \right| \leq \left| \exp(\beta(x) - \beta(y)) - 1 \right| \left| P - Q \right|, \]

then, according to [2] and [3], each value of \( M \) has range all of \( G \) and is invertible. This paper provides an alternate method for obtaining \( M(x, y)^{-1} \).

**References**


GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GEORGIA 30332