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A PRODUCT INTEGRAL REPRESENTATION FOR AN EVOLUTION SYSTEM

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ABSTRACT. This paper provides a product integral representation for a nonlinear evolution system. The representation is valid for expansive systems and provides an analysis in the nonexpansive case which is different from ones previously discovered.

In [7], D. Rutledge obtains a product integral representation for a nonexpansive, nonlinear semigroup. In [6], Neuberger gets such a representation for expansive semigroups by first considering nonexpansive evolution systems. This paper obtains a product integral representation for an expansive evolution system M. In this development, it is not required that $\lim_{h\to 0} h^{-1}[M(h, 0)-1]P$ exist. As a corollary to Theorem 3, a statement equivalent to the statement that M is nonexpansive is found.

Suppose that $\{G, +, |\cdot|\}$ is a complete, normed, Abelian group and that S is the set of real numbers. If f is a function from S to G and $a>b$, then denote the range of the restriction of f to $[b, a]$ by $f([b, a])$. Also, the statement that $\{s_p\}_0^n$ is a subdivision of $\{a, b\}$ means that s is a decreasing sequence with $s(0)=a$ and $s(n)=b$. The statement that t is a refinement of the subdivision s means that t is a subdivision of $\{a, b\}$ and that there is an increasing sequence u so that $s(p)=t(u(p))$ for $1\leqq p\leqq n$. Finally, if $\{f_p\}_1^n$ is a sequence of functions from G to G and g is in G, then

$$\left[\prod_{p=1}^{n} f_p\right](g) = f_1(f_2(\cdots f_n(g))).$$

An evolution system on G is a function M with domain contained in $S\times S$ so that if $x\geqq y$ then $M(x, y)$ is a function from G to G having the following properties:

(1) if $x\geqq y\geqq z$ then $M(x, y)M(y, z)=M(x, z)$ and $M(x, x)=1$, the identity function on G, and

(2) if t is a number and P is in G then the function g given by $g(x) = M(x, t)P$, for all $x\geqq t$, is continuous.

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In order to obtain a product integral representation for the evolution system M, two additional conditions are used:

(3) there is an increasing, continuous function β and a subset D of G so that

(a) if P is in D and $x>y$ then $M(x, y)P$ is in D, and

(b) if P is in D, $\epsilon>0$, $a>b$, and Q is in $M([b, a], b)P$ then there is a positive number δ so that if R is in $M([b, a], b)P$, $|Q-R|<\delta$, and $a \geqq x \geqq y \geqq b$, then

$$|[M(x, y) - 1]R - [M(x, y) - 1]Q| \leqq [\exp(\beta(x) - \beta(y)) - 1]\cdot\epsilon,$$

and

(4) there is a nondecreasing, continuous function α so that if $x>y$ and $\exp(\alpha(x)-\alpha(y))<2$, then $2-M(x, y)$ has range all of G and, if P and Q are in G, then

$$[2 - \exp(\alpha(x) - \alpha(y))]\cdot|P - Q| \leqq |[2 - M(x, y)]P - [2 - M(x, y)]Q|.$$

REMARK. It follows from condition (4) that if $\exp(\alpha(x)-\alpha(y))<2$, then $[2-M(x, y)]^{-1}$ has domain all of G, and if P and Q are in G then

$$|[2 - M(x, y)]^{-1}P - [2 - M(x, y)]^{-1}Q| \leqq [2 - \exp(\alpha(x) - \alpha(y))]^{-1}|P - Q|.$$

In this paper, the following three theorems are proved.

THEOREM 1. *Suppose that P is in D, $a>b$, and M satisfies conditions* (1)–(4). *It follows that $M(a, b)P={}_a\prod^b[2-M]^{-1}P$—in the sense that if $\epsilon>0$, then there is a subdivision s of $\{a, b\}$ so that if $\{t_p\}_0^n$ is a refinement of s then*

$$\left|M(a, b)P - \prod_{p=1}^{n}[2 - M(t_{p-1}, t_p)]^{-1}P\right| < \epsilon.$$

THEOREM 2. *Suppose that M satisfies conditions* (1)–(4), *if $x>y$ then $M(x, y)$ is continuous from G to G, D is dense in G, $a>b$, and P is in G, it follows that $M(a, b)P={}_a\prod^b[2-M]^{-1}P$.*

THEOREM 3. *Suppose that G is a Banach space, M satisfies conditions* (1)–(3). *If $x>y$ then $M(x, y)$ is continuous from G to G, D is dense in G, and ρ is a continuous, real valued function which is of bounded variation on each interval. These are equivalent*:

(a) *If $x>y$ and P and Q are in G then*

$$|M(x, y)P - M(x, y)Q| \leqq \exp(\rho(x) - \rho(y))\cdot|P - Q|.$$

(b) *If* $x>y$ *and* $\exp(\rho(x)-\rho(y))<2$, *then* $2-M(x, y)$ *has range all of* G *and, if* P *and* Q *are in* G, *then*

$$[2-\exp(\rho(x)-\rho(y))]\cdot|P-Q| \leqq |[2-M(x, y)]P-[2-M(x, y)]Q|.$$

INDICATION OF PROOFS. The following inequality is important in what follows; it may be established after considering the polynomial $P(z)=1-2z^2+z^3$. It is labeled Lemma 1 for later reference.

LEMMA 1. *If* x *is a number and* $1\leqq x\leqq(1+\sqrt{5})/2$ *then* $[2-x]^{-1}\leqq x^2$.

In the definitions and lemmas which follow, suppose that M satisfies conditions (1)–(4), $a>b$, and $\epsilon>0$.

DEFINITION. Define functions δ and B as follows: if P is in D and $a\geqq z\geqq b$ then $\delta(z, P)$ is the largest number d not exceeding 1 so that if Q is in $M([z, a], z)\ P$, $|Q-R|<d$, and $a\geqq x\geqq y\geqq z$ then

$$|[M(x, y)-1]Q-[M(x, y)-1]P|\leqq[\exp(\beta(x)-\beta(y))-1]\cdot\epsilon.$$

Also, $B(z, P)$ is the largest number u not exceeding a so that if $u>v>z$ then $|M(v, z)P-P|<\delta(z, P)$.

REMARK. Note that the existence of δ follows from condition (3) and of B follows from condition (2).

LEMMA 2. *Suppose that* P *is in* D. *If* $a\geqq x\geqq b$, $\{t_p\}_0^n$ *is a subdivision of* $\{B(x, P), x\}$, *and* j *is an integer in* $[1, n]$, *then*

$$|[M(t_{j-1}, t_j)-1]M(t_j, t_n)P-[M(t_{j-1}, t_j)-1]P| \leqq [\exp(\beta(t_{j-1})-\beta(t_j))-1]\cdot\epsilon.$$

INDICATION OF PROOF. If $\{t_p\}_0^n$ is a subdivision of $\{B(x, P), x\}$ and j is an integer in $[1, n]$ then $x\leqq t_j<B(x, P)$. Thus $|M(t_j, x)P-P|<\delta(x, P)$. Now, $M(t_j, x)P$ is in $M([x, a], x)P$, so if $a\geqq u\geqq v\geqq x$ then

$$|[M(u, v)-1]M(t_j, x)P-[M(u, v)-1]P| \leqq [\exp(\beta(u)-\beta(v))-1]\cdot\epsilon.$$

LEMMA 3. *Suppose that* P *is in* D, $\{t_p\}_0^\infty$ *is an increasing sequence with values in* $[b, a]$ *and limit* z. *There is a positive integer* N *so that if* $n>N$ *then* $B(t_n, M(t_n, b)P)\geqq z$.

INDICATION OF PROOF. Suppose that P is in D and t is an infinite increasing sequence with values in $[b, a]$ and limit z. The fact that $\{M(t_p, b)P\}_{p=0}^\infty$ converges in G and has limit $M(z, b)P$ follows from condition (2). Let Q be $M(z, b)P$. Since Q is in $M([b, a], b)P$, there is a number d so that $0<d<1$ and, if $|R-Q|<d$ and R is in $M([b, a], b)P$ and $a \geqq x \geqq y \geqq b$, then

$$|[M(x, y)-1]Q-[M(x, y)-1]R| \leqq [\exp(\beta(x)-\beta(y))-1]\cdot \epsilon/2.$$

Let w be so that if $z \geqq u \geqq w$ then $|Q-M(u, b)P|<d/4$. Let n be so that $t_n>w$. First, $\delta(t_n, M(t_n, b)P) \geqq d/2$ because: suppose R is in $M([t_n, a], b)P$ and $|R-M(t_n, b)P|<d/2$. Then $|R-Q|<d$ so that if $a \geqq x \geqq y \geqq b$ then

$$\begin{aligned} |[M(x, y) - 1]M(t_n, b)P - [M(x, y) - 1]R| \\ \leqq [\exp(\beta(x) - \beta(y)) - 1]\cdot[\epsilon/2 + \epsilon/2]. \end{aligned}$$

Finally, $B(t_n, M(t_n, b)P) \geqq z$ because: suppose that $t_n \leqq v \leqq z$. Then

$$\begin{aligned} |M(v, t_n)M(t_n, b)P - M(t_n, b)P| &\leqq |M(v, t_n)M(t_n, b)P - Q| \\ &+ |Q - M(t_n, b)P| \leqq d/4 + d/4 \leqq \delta(t_n, M(t_n, b)P). \end{aligned}$$

LEMMA 4. *Suppose that P is in D. There is a subdivision u of $\{a, b\}$ so that if $\{t_p\}_0^n$ is a refinement of u and p is an integer in $[1, n]$ then*

$$\begin{aligned} |[M(t_{p-1}, t_p) - 1]M(t_{p-1}, b)P - [M(t_{p-1}, t_p) - 1]M(t_p, b)P| \\ \leqq [\exp(\beta(t_{p-1}) - \beta(t_p)) - 1]\cdot 2\epsilon. \end{aligned}$$

INDICATION OF PROOF. Suppose that P is in D. By the previous lemma, there is a subdivision $\{u_q\}_0^m$ of $\{a, b\}$ so that if q is an integer in $[1, m]$ then $u_{q-1}=B(u_q, M(u_q, b)P)$. Let $\{t_p\}_0^n$ be a refinement of u and p be an integer in $[1, n]$. Let q be an integer in $[1, m]$ so that $u_{q-1} \geqq t_{p-1}>t_p \geqq u_q$. Then $|M(t_{p-1}, b)P-M(u_q, b)P|<\delta(u_q, M(u_q, b)P)$ and $|M(t_p, b)P-M(u_q, b)P|<\delta(u_q, M(u_q, b)P)$. Hence, if $a \geqq x \geqq y \geqq u_q$, then

$$\begin{aligned} |[M(x, y) - 1]M(t_{p-1}, b)P - [M(x, y) - 1]M(t_p, b)P| \\ \leqq [\exp(\beta(x) - \beta(y)) - 1]\cdot 2\epsilon. \end{aligned}$$

INDICATION OF PROOF OF THEOREM 1. Suppose that P is in D. Let u be a subdivision of $\{a, b\}$ as indicated in Lemma 4, $\{s_p\}_0^m$ be a refinement of u so that if p is an integer in $[1, m]$ then $\exp(\alpha(s_{p-1})-\alpha(s_p)) <(1+\sqrt{5})/2$, and $\{t_p\}_0^n$ be a refinement of s. By Lemma 1, if p is an integer in $[1, n]$ and P and Q are in G, then

$$\begin{aligned} |[2 - M(t_{p-1}, t_p)]^{-1}P - [2 - M(t_{p-1}, t_p)]^{-1}Q| \\ \leqq \exp(2[\alpha(t_{p-1}) - \alpha(t_p)])\cdot|P - Q|. \end{aligned}$$

$$\left| \prod_{p=1}^{n} [2 - M(t_{p-1}, t_p)]^{-1}P - M(a, b)P \right|$$

$$= \left| \sum_{j=1}^{n} \left\{ \prod_{p=1}^{n+1-j} [2 - M(t_{p-1}, t_p)]^{-1} M(t_{n+1-j}, b)P - \prod_{p=1}^{n-j} [2 - M(t_{p-1}, t_p)]^{-1} M(t_{n-j}, b)P \right\} \right|$$

$$\leqq \sum_{j=1}^{n} \exp(2[\alpha(a) - \alpha(t_{n+1-j})]) \cdot | M(t_{n+1-j}, b)P - [2 - M(t_{n-j}, t_{n+1-j})]M(t_{n-j}, b)P |$$

$$= \sum_{j=1}^{n} \exp(2[\alpha(a) - \alpha(t_{n+1-j})]) \cdot | [M(t_{n-j}, t_{n+1-j}) - 1]M(t_{n-j}, b)P - [M(t_{n-j}, t_{n+1-j}) - 1]M(t_{n+1-j}, b)P |$$

$$\leqq \sum_{j=1}^{n} [\exp(2[\alpha(a) - \alpha(t_{n+1-j})]) \cdot [\exp(\beta(t_{n-j}) - \beta(t_{n+1-j})) - 1] \cdot 2\epsilon$$

$$\leqq \exp(2[\alpha(a) - \alpha(b)]) \cdot [\exp(\beta(a) - \beta(b)) - 1] \cdot 2\epsilon.$$

To see this last inequality, one should note Lemma 2.2 of [4].

INDICATION OF PROOF OF THEOREM 2. Suppose that P and Q are in G, $a > b$, and $\{t_p\}_0^n$ is a subdivision of $\{a, b\}$ so that, if p is an integer in $[1, n]$, then $[2 - M(t_{p-1}, t_p)]^{-1}$ has domain all of G.

$$\left| M(a, b)P - \prod_{p=1}^{n} [2 - M(t_{p-1}, t_p)]^{-1}P \right| \leqq | M(a, b)P - M(a, b)Q |$$

$$+ \left| \prod_{p=1}^{n} [2 - M(t_{p-1}, t_p)]^{-1}Q - \prod_{p=1}^{n} [2 - M(t_{p-1}, t_p)]^{-1}P \right|$$

$$+ \left| \prod_{p=1}^{n} [2 - M(t_{p-1}, t_p)]^{-1}Q - M(a, b)Q \right|.$$

Thus, if D is dense in G and $M(a, b)$ is continuous from G to G, it follows from Lemma 1 that $M(a, b)P = {}_a\prod^b [2 - M]^{-1}P$.

LEMMA 5. *If ρ is a continuous function from S to S and is of bounded variation on each interval of S, $a > b$, and $\epsilon > 0$, then there is a subdivision s of $\{a, b\}$ so that if $\{t_p\}_0^n$ is a refinement of s then*

$$\left|\exp(\rho(a)-\rho(b)) - \prod_{p=1}^{n}[2-\exp(\rho(t_{p-1})-\rho(t_p))]^{-1}\right| < \epsilon.$$

INDICATION OF PROOF. Notice that if ρ is continuous and of bounded variation on each interval of S, $a>b$, and $\{t_p\}_0^n$ is a subdivision of $\{a, b\}$ so that, if p is an integer in $[1, n]$, then $\exp(\rho(t_{p-1})-\rho(t_p))<2$ then

$$\prod_{p=1}^{n}[2-\exp(\rho(t_{p-1})-\rho(t_p))]^{-1} \leqq \prod_{p=1}^{n}\left[2-\exp\left(\int_{t_p}^{t_{p-1}}|d\rho|\right)\right]^{-1}.$$

With techniques similar to those used in the proof of Theorem 1, it can be shown that, if

$$\exp\left(\int_{t_p}^{t_{p-1}}|d\rho|\right) < \frac{1+\sqrt{5}}{2} \qquad \text{for } p = 1, 2, \cdots, n,$$

then

$$\left|\prod_{p=1}^{n}[2-\exp(\rho(t_{p-1})-\rho(t_p))]^{-1} - \exp(\rho(a)-\rho(b))\right|$$
$$\leqq \exp\left(3\int_a^b |d\rho|\right)\cdot \sum_{j=1}^{n} \left| [\exp(\rho(t_{n-j})-\rho(t_{n+1-j}))-1]^2\right| .$$

The conclusion of the lemma follows.

INDICATION OF PROOF OF THEOREM 3. Suppose that G is a Banach space and that ρ is a function from S to S which is continuous and of bounded variation on each interval of S. Suppose also that $x>y$ and that $M(x, y)$ is a function from G to G having the property that if P and Q are in G then $|M(x, y)P-M(x, y)Q| \leqq \exp(\rho(x)-\rho(y))|P-Q| <2|P-Q|$. As in Lemma 1 of [5], let X be in G and $K(Z)$ be $.5[X+M(x, y)Z]$ for each Z in G. Then K is a contraction mapping and there is only one member Z of G so that $2Z-M(x, y)Z=X$. Furthermore, if P and Q are in G, then

$$|Q-P| \leqq .5|[2-M(x, y)]Q-[2-M(x, y)]P|$$
$$+ .5\exp(\rho(x)-\rho(y))|P-Q| .$$

Consequently, in Theorem 3, statement (a) implies statement (b). Finally, with G and ρ as supposed above, if M satisfies conditions (1)–(3), D is dense in G, statement (b) of Theorem 3 holds, and $x>y$, then, by Theorem 2, $M(x, y)P={}_x\prod^y [2-M]^{-1}P$ for each P in G and, by Lemma 5,

$$\left| {}_x\prod^y [2 - M]^{-1}P - {}_x\prod^y[2 - M]^{-1}Q \right| \leqq \exp(\rho(x) - \rho(y)) \mid P - Q \mid .$$

This completes the proof of Theorem 3.

Examples.

EXAMPLE 1. Let G be a Banach space and T be a one-parameter semigroup of nonlinear transformations on G. That is, T is a function from $[0, \infty)$ to the set of continuous transformations from G to G which satisfies

(1) $T(x)T(y) = T(x+y)$ if $x, y \geqq 0$,

(2) if P is in G and $g_p(x) = T(x)P$ for all x in $[0, \infty)$ then g_p is continuous and $\lim_{x\to 0^+} g_p(x) = P$,

(3) $|T(x)P - T(x)Q| \leqq |P-Q|$ if $x \geqq 0$ and P and Q are in G, and

(4) there is a dense subset D of G such that if P is in D then g'_p is continuous with domain $[0, \infty)$. By Theorem 2, if P is in D and $x > 0$, then $T(x)P = {}_x\prod^0[2 - T(-dI)]^{-1}P$. Compare [5] and Theorem 2 of [7].

EXAMPLE 2. Let f be an increasing function from the real numbers onto the real numbers so that f' is continuous and nonincreasing. Suppose also that g is increasing and continuous, and that, for $x > y$ and P a real number,

$$M(x, y)P = f(g(x) - g(y) + f^{-1}(P)).$$

M satisfies (1)–(4) but $\lim_{h\to 0^+} h^{-1}[M(h, 0) - 1]P$ may not exist. Compare Example 2 of [8], Example 3.4 of [1], and Theorem A of [6].

EXAMPLE 3. In case M satisfies conditions (1) and (2) and if P and Q are in G and $x > y$, then $|[M(x, y) - 1]P - [M(x, y) - 1]Q| \leqq [\exp(\beta(x) - \beta(y)) - 1]|P - Q|$, then, according to [2] and [3], each value of M has range all of G and is invertible. This paper provides an alternate method for obtaining $M(x, y)^{-1}$.

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