PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 3, March 1971

SOME REMARKS ON INJECTIVE ENVELOPES

D. J. H. GARLING

ABSTRACT. Another proof is given of the fact that every metric space (respectively every Banach space) has an injective envelope.

We shall be concerned with two categories of spaces and morphisms: the category C_1 of metric spaces and contraction mappings, and the category C_2 of (real or complex) Banach spaces and linear contraction mappings. We recall that a space E is said to be *injective* if whenever T is a morphism from a subspace Y of a space X into E, T can be extended to a morphism from X into E. A pair (i, Z) is said to be an *injective envelope* of Y if i is a (linear) isometry of Y into an injective space Z and if there is no proper subspace of Z containing $i(Y)$ which is injective ([2], [3], [4]). Every space in the two cateories mentioned above has an injective envelope ([2], [4], [6]): the purpose of this note is to give a proof valid in either category.

First we note that a space Y in either category can always be (linearly) isometrically embedded in an injective space Z. If Y is a metric space, for each x in Y and each integer n let $f_{x,n}(y) = \min(d(x, y), n)$, let $K = \{f_{x,n}: x \in Y, n \in N\}$ and let $B(K)$ denote the space of bounded real-valued functions on K with the usual metric. The evaluation map is then an isometric embedding of Y in the injective space $B(K)$. If Y is a Banach space, Y can be embedded isometrically and linearly in the injective space $B(S)$ of all bounded functions on the unit ball S of the dual of Y.

We need one further concept: if Y is a subspace of a space X, X is said to be an *essential extension* of Y [5] if whenever T is a morphism of X into a space W whose restriction to Y is an isometry, then T is an isometry. (It is easy to see that in the category C_2 this concept is the same as the concept of *bound extension* introduced by Kaufman [6].) Note that if X is an essential extension of Y and if X is injective then (i, X) is an injective envelope of Y (where i is the inclusion mapping).

Now let i be an isometric (linear) embedding of a space Y into an injective space Z, and let E denote the collection of subspace of Z which contain $i(Y)$ and are essential extensions of $i(Y)$. E may be

Received by the editors February 6, 1970 and, in revised form, April 20, 1970.

AMS 1969 *subject classifications*. Primary 4610, 5435.

Key words and phrases. Injective envelope, Banach space, metric space, contraction mapping.

partially ordered by inclusion. If $\{C_\alpha\}$ is a chain in E, let C be the closure of $U_\alpha C_\alpha$; it is clear that $C\in E$, and so Zorn's lemma ensures that E possesses maximal elements. We shall show that any such maximal element M is injective, so that (i, M) is an injective envelope of Y.

It is clearly sufficient to show that M is a retract of Z—that is, that there is a morphism P of Z onto M whose restriction to M is the identity. Let d denote the metric on Z and let S denote the collection of semimetrics s on Z satisfying:

(i) $s(i(y_1), i(y_2)) = d(i(y_1), i(y_2))$ for all y_1, y_2 in Y,

(ii) $s(z_1, z_2) \leqq d(z_1, z_2)$ for all z_1, z_2 in Z, (and

(iii) $s(z, 0)$ is a seminorm on Z).

An application of Zorn's lemma shows that under the natural ordering S has at least one minimal element s_0. Let R be the equivalence relation defined by s_0 (xRy if and only if $s_0(x, y) = 0$), and let q be the quotient mapping of Z into the completion K of the quotient space Z/R. q is clearly a morphism. Since qi is a (linear) isometry, $q|_M$ is a (linear) isometry of M into K. Since Z is injective, there is a morphism T of K into Z such that $Tq(m) = m$ for all m in M. Let L denote the closure of $T(K)$. Suppose that U is any morphism of L into a space (X, ρ) such that Ui is an isometry. Let $s_1(z_1,z_2) = \rho(UTq(z_1), UTq(z_2))$. Then $s_1 \in S$ and $s_1 \leqq s_0$; the minimality of s_0 implies that $s_1 = s_0$, and so U is an isometry. Thus L is an essential extension of $i(Y)$. But $L \supseteq T(K) \supseteq M$, and M is maximal, so that $T(K) = M$. Thus Tq is the required morphism.

As an immediate corollary it follows that if Y is a subspace of X, X is an injective envelope of Y if and only if X is injective and an essential extension of Y ([3], [5]).

Let us make three remarks on this proof. First, in the category C_2, the proof is very similar to the proof of [6, Theorem 1] given by Kaufman. Note, though, that Kaufman's proof is essentially linear. Secondly, the proof is only an existence proof. Isbell [4] has given an explicit construction for an injective envelope in the category C_1. Thirdly, the proof given is applicable to the category C_3 of compact metric spaces and contraction mappings, provided we know that a compact metric space can be isometrically embedded in an injective compact space. The easiest proof of this, however, seems to be that given by Isbell [4], who showed that an injective envelope of a compact metric space (in the category C_1) is compact.

I am greatly indebted to the referee, who suggested some important simplifications.

Since this paper was written, Mr. P. D. Bacsich has informed me that he also has found the proof given above, and Professor B. Banaschewski has informed me that his proof of the proposition in [1] is similar to that given above.

References

1. B. Banaschewski, *Essential extensions and injectivity for metric and Banach spaces*, Canad. Math. Bull. 12 (1969), 116–118.

2. H. B. Cohen, *Injective envelopes of Banach spaces*, Bull. Amer. Math. Soc. **70** (1964), 723–726. MR **32** #1536.

3. J. R. Isbell, *Injective envelopes of Banach spaces are rigidly attached*, Bull. Amer. Math. Soc. **70** (1964), 727–729. MR **32**# 1537.

4. ———, *Six theorems about injective metric spaces*, Comment. Math. Helv. **39** (1964), 65–76, MR **32** #431.

5. ———, *Three remarks on injective envelopes of Banach spaces*, J. Math. Anal. Appl. **27** (1969), 516–518.

6. R. Kaufman, *A type of extension of Banach spaces*, Acta. Sci. Math. (Szeged) **27** (1966), 163–166. MR **34** #4872.

St. John's College, University of Cambridge, Cambridge, England

Lehigh University, Bethlehem, Pennsylvania 18015