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CONCORDANT MAPPINGS AND THE CONCORDANT-DISSONANT FACTORIZATION OF AN ARBITRARY CONTINUOUS FUNCTION

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ABSTRACT. The property of concordance (weaker than monotonicity) is introduced, and a characterization of concordant mappings using quotient spaces enables the derivation of a new factorization, the concordant-dissonant factorization of an arbitrary continuous function.

1. **Introduction.** The present paper includes and extends results presented to the International Congress of Mathematicians in Moscow (1966).

§2 introduces concordant mappings and characterizes those which are quasi-compact (in the sense of Whyburn [**3**]) by the existence of a certain homeomorphism. The characterization allows us in §3 to obtain a new factorization, the concordant-dissonant factorization, of any continuous function whose domain and range may be arbitrary topological spaces.

§4 introduces quasi-concordant mappings on which we state one result, and which in our view deserve more attention.

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2. **Concordant mappings.** We begin by recalling Whyburn's definition [**3**] that a continuous surjective function f is *quasi-compact* provided the image under f of every open inverse set under f is open. That is, a continuous function f on a topological space X onto a topological space Y is quasi-compact if given any open $X' \subset X$ such that $X' = f^{-1}(Y')$ for some $Y' \subset Y$, then Y' is an open subset of Y. We note that open and closed surjective functions are quasi-compact and state three lemmas, the first of which is due to Whyburn [**3**].

LEMMA 1. *Suppose g_i on a topological space X to a topological space Y_i is a continuous surjective quasi-compact function for $i=1, 2$. If there*

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exists a bijection h on Y_1 to Y_2 such that $g_1 = h^{-1} \circ g_2$, $g_2 = h \circ g_1$, then h is a homeomorphism.

LEMMA 2. *The composition $f = h \circ g$ of two quasi-compact functions g on a topological space X onto a topological space Y and h on Y onto a topological space Z is quasi-compact.*

LEMMA 3. *Suppose that f is a continuous function on X to Y and that A is contained in a quasi-component of X. Then f(A) is contained in a quasi-component of Y.*

Let $\mathfrak{Q}_X$ denote the topological space whose points are the quasi-components of a topological space X and whose topology is the quotient topology derived from that of X. Suppose that f is a continuous function on X to Y. Then the last easily proved lemma allows us to define, in a natural way, an induced function $f_{\mathfrak{Q}}$ on $\mathfrak{Q}_X$ to $\mathfrak{Q}_Y$. Specifically, if Q_X is a quasi-component of X and if $f(Q_X)$ is contained in the quasi-component Q_Y of Y, we define $f_{\mathfrak{Q}}(Q_X) = Q_Y$. We note that an induced function $f_{\mathfrak{C}}$ on the space $\mathfrak{C}_X$ of components of X to the space $\mathfrak{C}_Y$ of components of Y may be defined in a similar manner.

A continuous function s on a topological space X to a topological space Y is said to be *concordant* if $s^{-1}(y)$ is contained in a quasi-component of X for each $y \in Y$. We are now in a position to state and prove the main result of this section.

THEOREM 4. *A necessary and sufficient condition for a surjective continuous quasi-compact function s on a topological space X to a topological space Y to be concordant is that the induced map $s_{\mathfrak{Q}}$ is a homeomorphism on $\mathfrak{Q}_X$ to $\mathfrak{Q}_Y$.*

PROOF. Sufficiency is obvious.

Let π_X denote the natural function on X to $\mathfrak{Q}_X$ and similarly define π_Y. Suppose there exists $Q_Y \in \mathfrak{Q}_Y$ such that $(s^{-1} \circ \pi_Y^{-1})Q_Y$ is not contained in a quasi-component of X; that is, there exists a partition $H|K$[1] of X with both $(s^{-1} \circ \pi_Y^{-1})Q_Y \cap H$ and $(s^{-1} \circ \pi_Y^{-1})Q_Y \cap K$ nonempty. Since s is concordant we have that, for each $y \in Y$, $s^{-1}(y)$ is contained wholly in H or wholly in K. Hence $\pi_Y^{-1}(Q_Y)$ is the disjoint union of two sets H', K' such that $s^{-1}(H') \subset H$ and $s^{-1}(K') \subset K$. But, since s is quasi-compact, $s(H)|s(K)$ is a partition of Y (s is concordant), and hence $\pi_Y^{-1}(Q_Y)$ cannot be a quasi-component. Lemma 3 now allows us to conclude that $s_{\mathfrak{Q}}$ is a bijection.

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[1] A *partition* $H|K$ of a topological space X is a pair H, K of nonempty disjoint open subsets of X whose union is X.

π_X, π_Y, being natural functions to quotient spaces, are quasi-compact, and $\pi_Y \circ s = s_Q \circ \pi_X$. Hence s_Q is continuous, its required openness is given by Lemma 2, and necessity is proved.

We recall that a continuous function f on a topological space X to a topological space Y is said to be *monotone* if $f^{-1}(y)$ is connected for all $y \in Y$. Clearly, if f is monotone it is also concordant.

3. The concordant-dissonant factorization. A *factorization of a function* f on a topological space X to a topological space Y is an (ordered) triple (s, t, M) where M is a topological space (the "*middle-space*" of the factorization), s is a surjective function on X to M, t is a function on M to Y, and $f = t \circ s$. A factorization (s, t, M) of f will be said to be *topologically-unique* if, whenever (s', t', M') is another factorization of f, there exists a homeomorphism h on M onto M' such that $h \circ s = s'$ and $t' \circ h = t$.

A continuous function t on a topological space M to a topological space Y is said to be *dissonant* if each quasi-component of M intersects $t^{-1}(y)$ in a degenerate set[2] for each $y \in Y$. A *concordant-dissonant factorization* of a function f on X to Y is a factorization (s, t, M) of f where s is concordant and t is dissonant.

THEOREM 5. *There exists a topologically-unique concordant-dissonant factorization of any continuous function with concordant factor quasi-compact.*

PROOF. Let f be a continuous function on X to Y and, as y varies over Y, consider the set $M = \{m\}$ of nonempty intersections of the quasi-components of X with $f^{-1}(y)$ to be a topological space, its topology being the quotient topology given by X. Let s be the natural map on X to M. Clearly, s is (surjective,) concordant, and quasi-compact. Define t on M to Y by $t = f \circ s^{-1}$. Since s is quasi-compact, clearly t is a (well-defined) continuous function on M to Y. Suppose t is not dissonant; that is, there exists $y \in Y$ and a quasi-component Q of M such that Q intersects $t^{-1}(y)$ in at least two distinct points $m_1, m_2 \in M$. As in the proof of Theorem 4, the inverse image under s of a quasi-component of M is contained in a quasi-component of X. Thus $s^{-1}(m_1)$, $s^{-1}(m_2)$ are contained in the same quasi-component of X as well as in $f^{-1}(y)$, contradicting the definition of M. Hence t is dissonant and we have demonstrated the existence of a concordant-dissonant factorization of f.

Now suppose there are two such factorizations (s, t, M) and (s', t', M'). If $m \in M$, the fact that s is concordant implies that $s^{-1}(m)$

[2] A *degenerate set* is a set which is either empty or a singleton.

is contained in a quasi-component Q of X. In fact

$$s^{-1}(m) = Q \cap ((f^{-1} \circ t)(m)) = Q',$$

say. For if $s^{-1}(m)$ were properly contained in Q', the singleton $\{m\}$ would be properly contained in $s(Q')$, and, using Lemma 3, we contradict the dissonance of t. Noting that $(t' \circ s' \circ s^{-1})(m) = t(m)$, similar arguments prove that $h = s' \circ s^{-1}$ is single-valued. Likewise $h' = s \circ (s')^{-1}$ is single-valued, and h' is clearly the inverse of h. Topological-uniqueness and our theorem now follow from Lemma 1.

We recall that a continuous function f on a topological space X to a topological space Y is said to be *light* if $f^{-1}(y)$ is totally disconnected for all $y \in Y$. Clearly, if f is dissonant it is also light. A *monotone-light factorization* of a function f on X to Y is a factorization (g, h, M) where g is monotone and h is light.

As in Theorem 5 we may deduce the following theorem which has been noted and used by Michael [1].

THEOREM 6. *There exists a topologically-unique monotone-light factorization of any continuous function with monotone factor quasi-compact provided that the range of the function is a T_1-space.*

We note that the hypothesis that the range of the functon in Theorem 6 be a T_1-space is needed to prove that there exists a monotone-light factorization of the function but is not needed to prove topological-uniqueness.

The fact that any continuous function on a compact space to a T_1-space has compact point inverses allows the immediate deduction of the following famous result of Whyburn [2].

COROLLARY 7. *Any continuous surjective function on a compact metric space X to a compact metric space Y has a topologically-unique factorization (g, h, M) such that g is monotone and such that $g^{-1}(m)$ is compact for all $m \in M$ and h is light.*[3]

4. Quasi-concordant mappings. A continuous function f on a topological space X to a topological space Y is said to be *quasi-concordant* if $(f(Q) \cap R)$ is either empty or equal to R for any quasi-components Q of X and R of Y. The following result is not difficult to prove.

THEOREM 8. *Let f be a continuous function on a topological space X*

[3] Whyburn calls a continuous function f on X to Y *monotone* if $f^{-1}(y)$ is a continuum for all $y \in Y$. Clearly, if X is compact and singletons of Y are closed, the two definitions agree.

to a locally connected space Y such that $f^{-1}(y)$ is compact for each $y \in Y$. Then in order that f be quasi-concordant it is both necessary and sufficient that it map clopen sets to clopen sets.

To our mind, quasi-concordant maps require further study.

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