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$M_0(G)$ IS NOT A PRIME L-IDEAL OF MEASURES

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ABSTRACT. A technique of Hewitt and Zuckerman is used to show that if G is any locally compact abelian group with dual Γ, then there exist nonzero positive regular Borel measures μ, ν on G, each one of which is mutually singular with each measure ω whose Fourier-Stieltjes transform vanishes at infinity on Γ and such that the Fourier-Stieltjes transform of the convolution $\mu * \nu$ vanishes at infinity on Γ.

0. **Introduction.** $M(G)$ is the *Banach algebra of regular Borel measures on a nondiscrete LCA group G, and $M_0(G)$ is the ideal of those measures whose Fourier-Stieltjes transforms vanish at infinity on the dual Γ of G.

An L-subspace (band) in $M(G)$ is a closed subspace I such that if $\mu \in I$ and $\nu \in M(G)$ is absolutely continuous with respect to μ then $\nu \in I$. The set $I^{\perp} = \{\mu : \mu$ is singular with each $\nu \in I\}$ is called the complement of I. We write $\mu \perp \nu$ if μ and ν are mutally singular. Each L-subspace gives a direct sum decomposition $M(G) = I \oplus I^{\perp}$.

An L-ideal I in $M(G)$ is an ideal which is an L-subspace. An L-ideal I is *prime* if $I^{\perp}$ is a subalgebra. Lemma 0.1 shows $M_0(G)$ is an L-ideal.

We prove the following:

THEOREM. *Let G be a nondiscrete* LCA *group. Then $M_0(G)^{\perp}$ contains positive nonzero measures μ, ν such that $\mu * \nu \in M_0(G)$.*

COROLLARY. *$M_0(G)$ is not a prime L-ideal.*

COMMENTS. Prime L-ideals have been constructed by Raĭkov (see [1]), Simon [5], Varopoulos [8], and Šreĭder [6]. Taylor [7] has characterized prime L-ideals in terms of generalized characters. Hewitt and Zuckerman [3] have shown that $L^1(G)$ is not a prime L-ideal.

We use the notation of [4]. $\nu \ll \mu$ means ν is absolutely continuous with respect to μ.

LEMMA 0.1. *$M_0(G)$ is an L-ideal.*

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PROOF. Since $M_0(G)$ is obviously an ideal, we show $\mu \in M_0(G)$ and $\nu \ll \mu$ imply $\nu \in M_0(G)$.

For each $\epsilon > 0$, there exists an element $\hat{g} \in A(G)$, such that

(i) $g \in C_c(\Gamma)$,

(ii) if $d\omega = \hat{g}\, d\mu$, then $\|\omega - \nu\| < \epsilon$.

Let F be the support of g, and let $|\hat{\mu}(\gamma)| < \epsilon/\|g\|$ if $\gamma \notin E$ be a compact set.

Then $|\hat{\omega}(\gamma)| < 2\epsilon$ outside $E+F$, so $|\hat{\nu}(\gamma)| < 3\epsilon$ if $\gamma \notin E+F$. Q.E.D.

LEMMA 0.2 [6, p. 372]. *If μ, $\nu \in M(G)$ are positive measures and $\mu_1 \ll \mu$, $\nu_1 \ll \nu$, then $\mu_1 * \nu_1 \ll \mu * \nu$.*

1. **Proof of Theorem.** To prove the Theorem, we use a technique of Hewitt and Zuckerman:

DEFINITION 1.1. A subset Λ of a group Γ is *dissassociate* [3] if for any N and any choice $\epsilon_j \in \{-2, -1, 0, 1, 2\}$ and $\lambda_j \in \Lambda$ $(j = 1, \cdots, N)$,

$$\sum_{j=1}^{N} \epsilon_j \lambda_j = 0_\Gamma \Rightarrow \epsilon_1 \lambda_1 = \cdots = \epsilon_N \lambda_N = 0_\Gamma.$$

THEOREM 1.2 (HEWITT AND ZUCKERMAN [3, Theorems 3.2 and 4.4]). *Let G be either the circle group T or a compact 0-dimensional metric group. Then there is an infinite dissassociate subset $\{\lambda_j\}_1^\infty = \Lambda \subseteq \Gamma$ such that for any choice $\beta = \{\beta_j : j = 1, 2, \cdots\}$ of real numbers $|\beta_j| \leqq 1/2$ there exists a continuous positive measure $\mu \in M(G)$ with*

$$\text{(i)} \quad \begin{aligned} \hat{\mu}(\gamma) &= \prod_{n=1}^{N} \beta_{j_n} && \textit{if } \gamma = \sum_{n=1}^{N} \epsilon_{j_n} \lambda_{j_n},\ \epsilon_{j_n} = \pm 1, \\ &= 1 && \textit{if } \gamma = 0_\Gamma, \\ &= 0 && \textit{otherwise.} \end{aligned}$$

(ii) *$\mu \geqq 0$ and $\|\mu\| \leqq 1$.*

Furthermore

(iii) *μ is singular if $\sum |\beta_j|^2 = \infty$. Otherwise μ is absolutely continuous.*

COROLLARY 1.3. *Let G be either the group T or a compact abelian 0-dimensional metric group. Then $M(G)$ contains positive nonzero measures μ, ν such that*

(i) μ, $\nu \in M_0(G)^\perp$,

(ii) $\mu * \nu \in M_0(G)$.

PROOF. Let

$$\beta^{(1)} = \{\beta_j^{(1)}:\beta_{2j}^{(1)} = 1/j, \beta_{2j+1}^{(1)} = 1/2\},$$

$$\beta^{(2)} = \{\beta_j^{(2)}:\beta_{2j}^{(2)} = 1/2, \beta_{2j+1}^{(2)} = 1/j\}.$$

If μ_1 is the measure constructed by Theorem 1.2 for $\beta^{(1)}$ and ν_1 is that for $\beta^{(2)}$, then $\mu_1 * \nu_1$ is by (i) the measure from $\beta^{(3)} = \{\beta_j^{(1)}\beta_j^{(2)}\}$. Hence $\mu * \nu \in M_0(G)$.

Since μ_1, $\nu_1 \notin M_0(G)$, there exist positive measures $0 \neq \mu \ll \mu_1$, $0 \neq \nu \ll \nu_1$ such that μ, $\nu \in M_0(G)^\perp$. By Lemma 0.2, $\mu * \nu \ll \mu_1 * \nu_1$. Since $M_0(G)$ is an L-ideal, $\mu * \nu \in M_0(G)$. Q.E.D.

Corollary 1.4. *Let G be an compact abelian group. Then $M(G)$ contains positive nonzero measures μ and ν such that*

(i) μ, $\nu \in M_0(G)^\perp$,

(ii) $\mu * \nu \in M_0(G)$.

Proof [3]. Either Γ contains an element γ_0 of infinite order, or Γ contains an infinite countable torsion subgroup. In the first case, let Λ be the subgroup γ_0 generates; in the second case let Λ be the infinite torsion subgroup. Let $H = \{g \in G: (h, \lambda) = 1, \lambda \in \Lambda\}$. Then $\hat{\Lambda} = G/H$ and G/H satisfies the hypotheses of Theorem 1.6. Hence there are positive measures μ_1, $\nu_1 \in M_0(G/H)^\perp$ such that $\mu_1 * \nu_1 \in M_0(G/H)$.

Let ω be Haar measure on the compact group H. Then $G \to G/H$ induces an isomorphism π of the subalgebra $\omega * M(G)$ with $M(G/H)$. Let μ, $\nu \in \omega * M(G)$ such that $\mu \to \mu_1$ and $\nu \to \nu_1$. Since $\hat{\omega}(\gamma) = 1$ if $\gamma \in \Lambda$ and $\hat{\omega}(\gamma) = 0$ otherwise, we see that $\mu * \nu \in M_0(G)$.

On the other hand if $\sigma \ll \mu$, $\sigma \geqq \mu$, and $\sigma \in M_0(G)$, then $\pi\sigma \in M_0(G/H)$ and $\pi\sigma \ll \mu_1$. Hence, $\pi\sigma = 0$. Since $\sigma \geqq 0$, $\|\sigma\| = \|\pi\sigma\| = 0$ and $\sigma = 0$. Thus μ (and by the same argument ν) is an element of $M_0(G)^\perp$. Q.E.D.

We now use the structure theorem for locally compact abelian groups to extend Corollary 1.4 to the general group:

G may be written $G = \boldsymbol{R}^n \times D$ [2, p. 389] where D has a compact open subgroup and $n \geqq 0$.

We first suppose $n = 0$, and let C be the compact open subgroup of $D = G$. Then by 1.4, there are positive measures μ, ν concentrated on C which, *as elements of* C, satisfy μ, $\nu \in M_0(C)^\perp$ and $\mu * \nu \in M_0(G)$.

We claim μ, $\nu \in M_0(G)^\perp$ and $\mu * \nu \in M_0(G)$. The first assertion follows from the fact that $M(C)$ is an L-subalgebra of $M(G)$. The second follows from the fact that $(\mu * \nu)\hat{}$ is constant on the cosets of the compact subgroup Λ of Γ:

$$\Lambda = \{\gamma \in \Gamma: (x, \gamma) = 1 \text{ for all } x \in C\}.$$

We now suppose $n>0$, and that $\mu_1, \nu_1 \in M_0(C)^{\perp}$ and $\mu_1 * \nu_1 \in M_0(C)$.

Let μ_2, ν_2 be positive measures on the n-torus T^n such that μ_2, $\nu_2 \in M_0(T^n)^{\perp}$ and $\mu_2 * \nu_2 \in M_0(T^n)$. We will "lift" μ_2 and ν_2 to $\mathbf{R}^n$ obtaining μ_3 and ν_3 and show that μ_3, $\nu_3 \in M_0(\mathbf{R}^n)^{\perp}$, while $\mu_3 * \nu_3 \in M_0(\mathbf{R}^n)$. Lemmas 1.5 and 1.6 will then show that $\mu = \mu_3 \times \mu_1$ and $\nu = \nu_3 \times \nu_1 \in M_0(G)^{\perp}$, and $\mu * \nu \in M_0(G)$, so the proof of the theorem will be complete.

LEMMA 1.5. *Let $\mu \in M(G)$, $\nu \in M(H)$. Then*
(i) $\mu \in M_0(G)$ and $\nu \in M_0(H)$ imply $\mu \times \nu \in M_0(G \times H)$;
(ii) $\mu \in M_0(G)^{\perp}$ implies $\mu \times \nu \in M_0(G \times H)^{\perp}$.

PROOF. (i) The dual group of $G \times H$ is $\hat{G} \times \hat{H}$.

Suppose $|\hat{\mu}| < \epsilon$ outside a compact set $E \subseteq \hat{G}$ and $|\hat{\nu}| < \epsilon$ outside a compact set $F \subseteq \hat{H}$. Then

$$|\hat{\mu} \times \hat{\nu}(\gamma, \rho)| = |\hat{\mu}(\gamma)\hat{\nu}(\rho)| < \epsilon(\|\mu \times \nu\|)$$

if $(\gamma, \rho) \notin E \times F$. Hence $\mu \times \nu \in M_0(G \times H)$.

(ii) Let $\omega \in M_0(G \times H)$ where $\omega \ll \mu \times \nu$, say $d\omega = f(x, y)d\mu(x)d\nu(y)$, where f is a Borel function on $G \times H$. Since $M_0(G \times H)$ is an L-ideal, we may assume f is bounded. Then choose $(\gamma, \rho) \in \hat{G} \times \hat{H}$ so $\hat{\omega}(\gamma, \rho) \neq 0$.

$$\begin{aligned}\hat{\omega}(\gamma, \rho) &= \iint (x, \gamma)(y, \rho)f(x, y)d\mu(x)d\nu(y) \\ &= \iint (x, \gamma)(y, \rho)f(x, y)d\nu(y)d\mu(x) \\ &= \int (x, \gamma)F(x)d\mu(x),\end{aligned}$$

where the second line is a consequence of Fubini's Theorem and $F(x) = \int (y, \rho)f(x, y)d\nu(y)$. Then the measure $d\sigma = F(x)d\mu(x)$ is a nonzero element of $M_0(G)^{\perp}$. Hence $\hat{\sigma}(\gamma) \not\to 0$. Hence $\hat{\omega}(\gamma, \rho) \not\to 0$ as $(\gamma, \rho) \to \infty$. Q.E.D.

We now lift our measures: If $\omega \in M(T^n)$, define $\omega^{\#} \in M(\mathbf{R}^n)$ by

$$\omega^{\#}(E) = \omega(E \cap [0, 2\pi)^n)$$

for each Borel subset $E \subseteq \mathbf{R}^n$. ($[0, 2\pi)^n$ is identified with T^n by $(x_1, \cdots, x_n) \to (e^{ix_1}, \cdots, e^{ix_n})$.)

Let $P: M(\mathbf{R}^n) \to M(T^n)$ be the map induced by $\mathbf{R}^n \to \mathbf{R}^n/\mathbf{Z}^n = T^n$.

LEMMA 1.6. *If $\mu, \nu \in M(T^n)$ are positive measures; then*
(i) $P(\mu^{\#}) = \mu$;

(ii) $\mu \in M_0(T^n)^\perp$ *implies* $\mu^\# \in M_0(\mathbf{R}^n)^\perp$;
(iii) $\mu * \nu \in M_0(T^n)$ *implies* $\mu^\# * \nu^\# \in M_0(\mathbf{R}^n)$.

PROOF. (i) is obvious. Let $\nu \ll \mu^\#$. Then $P\nu \ll \mu$, so $P\nu \perp M_0(T^n)$ if $\mu \perp M_0(T^n)$. Hence $(P\nu)^\wedge$ does not vanish at infinity. Since $((P\nu)^\#)^\wedge = \hat{\nu}$ takes (on $\mathbf{R}^n$) values which include the values of $(P\nu)^\wedge$ on Z^n, $\nu \notin M_0(\mathbf{R}^n)$. Hence $\mu^\# \perp M_0(\mathbf{R}^n)$. Hence (ii) holds.

To prove (iii) first write each element $q = (q_1, \cdots, q_n) \in \mathbf{R}^n$ as

$$(q_1, \cdots, q_n) = (2\pi m_1 + r_1, \cdots, 2\pi m_n + r_n) = 2\pi m + r$$

where $r = (r_1, \cdots, r_n) \in [0, 2\pi)^n$ and $m \in Z^n$. Then

$$(\mu^\# * \nu^\#)^\wedge(q) = \hat{\mu}_r(m)\hat{\nu}_r(m),$$

where $d\mu_r(x) = \exp(2\pi i x \cdot r) d\mu(x)$ and $d\nu_r(x) = \exp(2\pi i x \cdot r) d\nu(x)$.

From the next lemma we see that $\mu^\# * \nu^\# \in M_0(\mathbf{R}^n)$, so Lemma 1.6 is proved.

LEMMA 1.7. *Let* $\mu, \nu \in M(T^n)$ *be positive measures, and suppose* $\mu * \nu \in M_0(T^n)$. *Then for each* $\delta > 0$ *there is a compact set* $E \subseteq \mathbf{Z}^n$ *such that* $m \notin E$ *and* $r \in [0, 2\pi)^n$ *imply* $|(\mu_r * \nu_r)^\wedge(m)| < \delta$.

PROOF. Since $\mu_r \ll \mu$ and $\nu_r \ll \nu$, Lemmas 0.1 and 0.2 imply $\mu_r * \nu_r \ll \mu * \nu$, and $\mu_r * \nu_r \in M_0(T^n)$. Let $F_r \subseteq \mathbf{Z}^n$ be such that $m \notin F_r$ implies $|(\mu_r * \nu_r)^\wedge(m)| < \delta/2$. The obvious norm-continuity of $r \to \mu_r * \nu_r$ and the compactness of $[0, 2\pi]^n$ show that for a union E of a finite number $F_{r_1}, \cdots, F_{r_n}$ of the F_r we have

$$m \notin E = F_{r_1} \cup \cdots \cup F_{r_n} \text{ implies } |(\mu_r * \nu_r)^\wedge(m)| < \delta$$

for all $r \in [0, 2\pi]^n$. Q.E.D.

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