$M_0(G)$ IS NOT A PRIME $L$-IDEAL OF MEASURES

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Abstract. A technique of Hewitt and Zuckerman is used to show that if $G$ is any locally compact abelian group with dual $\Gamma$, then there exist nonzero positive regular Borel measures $\mu, \nu$ on $G$, each one of which is mutually singular with each measure $\omega$ whose Fourier-Stieltjes transform vanishes at infinity on $\Gamma$ and such that the Fourier-Stieltjes transform of the convolution $\mu \ast \nu$ vanishes at infinity on $\Gamma$.

0. Introduction. $M(G)$ is the Banach algebra of regular Borel measures on a nondiscrete LCA group $G$, and $M_0(G)$ is the ideal of those measures whose Fourier-Stieltjes transforms vanish at infinity on the dual $\Gamma$ of $G$.

An $L$-subspace (band) in $M(G)$ is a closed subspace $I$ such that if $\mu \in I$ and $\nu \in M(G)$ is absolutely continuous with respect to $\mu$ then $\nu \in I$. The set $I^\perp = \{ \mu : \mu$ is singular with each $\nu \in I \}$ is called the complement of $I$. We write $\mu \perp \nu$ if $\mu$ and $\nu$ are mutually singular. Each $L$-subspace gives a direct sum decomposition $M(G) = I \oplus I^\perp$.

An $L$-ideal $I$ in $M(G)$ is an ideal which is an $L$-subspace. An $L$-ideal $I$ is prime if $I^\perp$ is a subalgebra. Lemma 0.1 shows $M_0(G)$ is an $L$-ideal.

We prove the following:

Theorem. Let $G$ be a nondiscrete LCA group. Then $M_0(G)^\perp$ contains positive nonzero measures $\mu, \nu$ such that $\mu \ast \nu \in M_0(G)$.

Corollary. $M_0(G)$ is not a prime $L$-ideal.

Comments. Prime $L$-ideals have been constructed by Raïkov (see [1]), Simon [5], Varopoulos [8], and Šreïder [6]. Taylor [7] has characterized prime $L$-ideals in terms of generalized characters. Hewitt and Zuckerman [3] have shown that $L^1(G)$ is not a prime $L$-ideal.

We use the notation of [4]. $\nu \ll \mu$ means $\nu$ is absolutely continuous with respect to $\mu$.

Lemma 0.1. $M_0(G)$ is an $L$-ideal.

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Proof. Since $M_0(G)$ is obviously an ideal, we show $\mu \in M_0(G)$ and $\nu \ll \mu$ imply $\nu \in M_0(G)$.

For each $\epsilon > 0$, there exists an element $g \in \mathbb{C}(G)$, such that

(i) $g \in C_c(T)$,

(ii) if $d\omega = gd\mu$, then $|\omega - \nu| < \epsilon$.

Let $F$ be the support of $g$, and let $|\mu(\gamma)| < \epsilon/\|g\|$ if $\gamma \notin E$ be a compact set.

Then $|\phi(\gamma)| < 2\epsilon$ outside $E + F$, so $|\phi(\gamma)| < 3\epsilon$ if $\gamma \in E + F$. Q.E.D.

Lemma 0.2 [6, p. 372]. If $\mu, \nu \in M(G)$ are positive measures and $\mu \ll \mu, \nu \ll \nu$, then $\mu \ast \nu \ll \mu \ast \nu$.

1. Proof of Theorem. To prove the Theorem, we use a technique of Hewitt and Zuckerman:

Definition 1.1. A subset $A$ of a group $\Gamma$ is dissassociate [3] if for any $N$ and any choice $e_3 \in \{-2, -1, 0, 1, 2\}$ and $\lambda_j \in \Lambda (j = 1, \ldots, N)$,

$$\sum_{j=1}^{N} e_3 \lambda_j = 0 \Rightarrow e_1 \lambda_1 = \cdots = \lambda_N = 0 \Gamma.$$

Theorem 1.2 (Hewitt and Zuckerman [3, Theorems 3.2 and 4.4]). Let $G$ be either the circle group $T$ or a compact 0-dimensional metric group. Then there is an infinite dissassociate subset $\{\lambda_j\}_{j=1}^\infty \subseteq \Gamma$ such that for any choice $\beta = \{\beta_j; j = 1, 2, \cdots\}$ of real numbers $|\beta_j| \leq 1/2$ there exists a continuous positive measure $\mu \in M(G)$ with

$$\mu(\gamma) = \prod_{n=1}^{N} \beta_j \quad \text{if} \quad \gamma = \sum_{n=1}^{N} e_j \lambda_n, \quad e_j = \pm 1,$$

(i) $= 1$ if $\gamma = 0 \Gamma$,

(ii) $= 0$ otherwise.

Furthermore

(iii) $\mu$ is singular if $\sum |\beta_j|^2 = \infty$. Otherwise $\mu$ is absolutely continuous.

Corollary 1.3. Let $G$ be either the group $T$ or a compact abelian 0-dimensional metric group. Then $M(G)$ contains positive nonzero measures $\mu, \nu$ such that

(i) $\mu, \nu \in M_0(G)^\perp$,

(ii) $\mu \ast \nu \in M_0(G)$.

Proof. Let
\[ \beta^{(1)} = \{ \beta_j^{(1)} : \beta_2^{(1)} = 1/j, \beta_{2j+1}^{(1)} = 1/2 \}, \]
\[ \beta^{(2)} = \{ \beta_j^{(2)} : \beta_2^{(2)} = 1/2, \beta_{2j+1}^{(2)} = 1/j \}. \]

If \( \mu_1 \) is the measure constructed by Theorem 1.2 for \( \beta^{(1)} \) and \( \nu_1 \) is that for \( \beta^{(2)} \), then \( \mu_1 \ast \nu_1 \) is by (i) the measure from \( \beta^{(3)} = \{ \beta_j^{(1)} \beta_j^{(2)} \} \). Hence \( \mu \ast \nu \in M_0(G) \).

Since \( \mu_1, \nu_1 \in M_0(G) \), there exist positive measures \( 0 \neq \mu \ll \mu_1, 0 \neq \nu \ll \nu_1 \) such that \( \mu, \nu \in M_0(G)^\perp \). By Lemma 0.2, \( \mu \ast \nu \ll \mu_1 \ast \nu_1 \). Since \( M_0(G) \) is an \( \mathcal{L} \)-ideal, \( \mu \ast \nu \in M_0(G) \). Q.E.D.

**Corollary 1.4.** Let \( G \) be an compact abelian group. Then \( M(G) \) contains positive nonzero measures \( \mu \) and \( \nu \) such that
(i) \( \mu, \nu \in M_0(G)^\perp \),
(ii) \( \mu \ast \nu \in M_0(G) \).

**Proof [3].** Either \( \Gamma \) contains an element \( \gamma_0 \) of infinite order, or \( \Gamma \) contains an infinite countable torsion subgroup. In the first case, let \( \Lambda \) be the subgroup \( \gamma_0 \) generates; in the second case let \( \Lambda \) be the infinite torsion subgroup. Let \( H = \{ h \in G : (h, \lambda) = 1, \lambda \in \Lambda \} \). Then \( \hat{\Lambda} = G/H \) and \( G/H \) satisfies the hypotheses of Theorem 1.6. Hence there are positive measures \( \mu_1, \nu_1 \in M_0(G/H)^\perp \) such that \( \mu_1 \ast \nu_1 \in M_0(G/H) \).

Let \( \omega \) be Haar measure on the compact group \( H \). Then \( G \rightarrow G/H \) induces an isomorphism \( \pi \) of the subalgebra \( \omega \ast M(G) \) with \( M(G/H) \). Let \( \mu, \nu \in \omega \ast M(G) \) such that \( \mu \rightarrow \mu_1 \) and \( \nu \rightarrow \nu_1 \). Since \( \omega(\gamma) = 1 \) if \( \gamma \in \Lambda \) and \( \omega(\gamma) = 0 \) otherwise, we see that \( \mu \ast \nu \in M_0(G) \).

On the other hand if \( \sigma \ll \mu, \sigma \geq \mu, \) and \( \sigma \in M_0(G) \), then \( \pi \sigma \in M_0(G/H) \) and \( \pi \sigma = 0 \). Since \( \sigma \geq 0, \| \sigma \| = \| \pi \sigma \| = 0 \) and \( \sigma = 0 \). Thus \( \mu \) (and by the same argument \( \nu \)) is an element of \( M_0(G)^\perp \). Q.E.D.

We now use the structure theorem for locally compact abelian groups to extend Corollary 1.4 to the general group:

\( G \) may be written \( G = R^n \times D \) [2, p. 389] where \( D \) has a compact open subgroup and \( n \geq 0 \).

We first suppose \( n = 0 \), and let \( C \) be the compact open subgroup of \( D = G \). Then by 1.4, there are positive measures \( \mu, \nu \) concentrated on \( C \), which, as elements of \( C \), satisfy \( \mu, \nu \in M_0(C)^\perp \) and \( \mu \ast \nu \in M_0(G) \).

We claim \( \mu, \nu \in M_0(G)^\perp \) and \( \mu \ast \nu \in M_0(G) \). The first assertion follows from the fact that \( M(C) \) is an \( \mathcal{L} \)-subalgebra of \( M(G) \). The second follows from the fact that \( (\mu \ast \nu)^\ast \) is constant on the cosets of the compact subgroup \( \Lambda \) of \( \Gamma \):

\[ \Lambda = \{ \gamma \in \Gamma : (x, \gamma) = 1 \text{ for all } x \in C \}. \]
We now suppose \( n > 0 \), and that \( \mu_1, \nu_1 \in M_0(C)^\perp \) and \( \mu_1 \ast \nu_1 \in M_0(C) \).

Let \( \mu_2, \nu_2 \) be positive measures on the \( n \)-torus \( T^n \) such that \( \mu_2, \nu_2 \in M_0(T^n)^\perp \) and \( \mu_2 \ast \nu_2 \in M_0(T^n) \). We will "lift" \( \mu_2 \) and \( \nu_2 \) to \( \mathbb{R}^n \) obtaining \( \mu_3 \) and \( \nu_3 \) and show that \( \mu_3, \nu_3 \in M_0(\mathbb{R}^n)^\perp \), while \( \mu_3 \ast \nu_3 \in M_0(\mathbb{R}^n) \). Lemmas 1.5 and 1.6 will then show that \( \mu = \mu_3 \times \mu_1 \) and \( \nu = \nu_3 \times \nu_1 \in M_0(G)^\perp \), and \( \mu \ast \nu \in M_0(G) \), so the proof of the theorem will be complete.

**Lemma 1.5.** Let \( \mu \in M(G), \nu \in M(H) \). Then

(i) \( \mu \in M_0(G) \) and \( \nu \in M_0(H) \) imply \( \mu \times \nu \in M_0(G \times H) \);

(ii) \( \mu \in M_0(G)^\perp \) implies \( \mu \times \nu \in M_0(G \times H)^\perp \).

**Proof.** (i) The dual group of \( G \times H \) is \( \hat{G} \times \hat{H} \).

Suppose \( |\hat{\mu}| < \epsilon \) outside a compact set \( E \subseteq \hat{G} \) and \( |\hat{\nu}| < \epsilon \) outside a compact set \( F \subseteq \hat{H} \).

Thus

\[
|\hat{\mu} \times \hat{\nu}(\gamma, \rho)| = |\hat{\mu}(\gamma)\hat{\nu}(\rho)| < \epsilon \langle ||\mu \times \nu|| \rangle
\]

if \( (\gamma, \rho) \notin E \times F \). Hence \( \mu \times \nu \in M_0(G \times H) \).

(ii) Let \( \omega \in M_0(G \times H) \) where \( \omega \ll \mu \times \nu \), say \( d\omega = f(x, y)d\mu(x)d\nu(y) \), where \( f \) is a Borel function on \( G \times H \). Since \( M_0(G \times H) \) is an \( L \)-ideal, we may assume \( f \) is bounded. Then choose \( (\gamma, \rho) \in \hat{G} \times \hat{H} \) so \( \omega(\gamma, \rho) \neq 0 \).

\[
\omega(\gamma, \rho) = \int \int (x, \gamma)(y, \rho)f(x, y)d\mu(x)d\nu(y)
\]

\[
= \int \int (x, \gamma)(y, \rho)f(x, y)d\nu(y)d\mu(x)
\]

\[
= \int (x, \gamma)F(x)d\mu(x),
\]

where the second line is a consequence of Fubini's Theorem and \( F(x) = \int (y, \rho)f(x, y)d\nu(y) \). Then the measure \( d\sigma = F(x)d\mu(x) \) is a nonzero element of \( M_0(G)^\perp \). Hence \( \omega(\gamma, \rho) \neq 0 \) as \( (\gamma, \rho) \to \infty \). Q.E.D.

We now lift our measures: If \( \omega \in M(T^n) \), define \( \omega^\sharp \in M(\mathbb{R}^n) \) by

\[
\omega^\sharp(E) = \omega(E \cap [0, 2\pi)^n)
\]

for each Borel subset \( E \subseteq \mathbb{R}^n \). \([0, 2\pi)^n \) is identified with \( T^n \) by \( (x_1, \ldots, x_n) \to (e^{ix_1}, \ldots, e^{ix_n}) \).

Let \( P: M(\mathbb{R}^n) \to M(T^n) \) be the map induced by \( \mathbb{R}^n \to \mathbb{R}^n / \mathbb{Z}^n = T^n \).

**Lemma 1.6.** If \( \mu, \nu \in M(T^n) \) are positive measures; then

(i) \( P(\mu^\flat) = \mu \);
(ii) \( \mu \in M_0(T^n)^\perp \) implies \( \mu^* \in M_0(\mathbb{R}^n)^\perp \);  
(iii) \( \mu * \nu \in M_0(T^n) \) implies \( \mu^* * \nu^* \in M_0(\mathbb{R}^n) \).

**Proof.** (i) is obvious. Let \( \nu \ll \mu \). Then \( P\nu \ll \mu \), so \( P\nu \perp M_0(T^n) \) if \( \mu \perp M_0(T^n) \). Hence \( (P\nu)^* \) does not vanish at infinity. Since \( ((P\nu)^*)^* = \rho \) takes (on \( \mathbb{R}^n \)) values which include the values of \( (P\nu)^* \) on \( \mathbb{Z}^n \), \( \nu \subseteq M_0(\mathbb{R}^n) \). Hence \( \mu^* \perp M_0(\mathbb{R}^n) \). Hence (ii) holds.

To prove (iii) first write each element \( q = (q_1, \ldots, q_n) \in \mathbb{R}^n \) as
\[
(q_1, \ldots, q_n) = (2\pi m_1 + r_1, \ldots, 2\pi m_n + r_n) = 2\pi m + r
\]
where \( r = (r_1, \ldots, r_n) \in [0, 2\pi)^n \) and \( m \in \mathbb{Z}^n \). Then
\[
(\mu^* * \nu^*)(q) = \mu_r(m) \nu_r(m),
\]
where \( d\mu_r(x) = \exp(2\pi ix \cdot r) d\mu(x) \) and \( d\nu_r(x) = \exp(2\pi ix \cdot r) d\nu(x) \).

From the next lemma we see that \( \mu^* * \nu^* \in M_0(\mathbb{R}^n) \), so Lemma 1.6 is proved.

**Lemma 1.7.** Let \( \mu, \nu \in M(T^n) \) be positive measures, and suppose \( \mu * \nu \in M_0(T^n) \). Then for each \( \delta > 0 \) there is a compact set \( E \subseteq \mathbb{Z}^n \) such that \( m \in E \) and \( r \in [0, 2\pi)^n \) imply \( |(\mu_r * \nu_r)^*(m)| < \delta \).

**Proof.** Since \( \mu_r \ll \mu \) and \( \nu_r \ll \nu \), Lemmas 0.1 and 0.2 imply \( \mu_r * \nu_r \ll \mu * \nu \), and \( \mu_r * \nu_r \in M_0(T^n) \). Let \( F_r \subseteq \mathbb{Z}^n \) be such that \( m \in F_r \) implies \( |(\mu_r * \nu_r)^*(m)| < \delta/2 \). The obvious norm-continuity of \( r \rightarrow \mu_r * \nu_r \), and the compactness of \( [0, 2\pi]^n \) show that for a union \( E \) of a finite number \( F_{r_1}, \ldots, F_{r_n} \) of the \( F_r \) we have
\[
m \notin E = F_{r_1} \cup \cdots \cup F_{r_n} \text{ implies } |(\mu_r * \nu_r)^*(m)| < \delta
\]
for all \( r \in [0, 2\pi]^n \). Q.E.D.

**References**

