PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 3, March 1971

CONVERSION OF THE PERMANENT INTO THE DETERMINANT[1]

P. M. GIBSON

ABSTRACT. Let A be an n-square (0, 1)-matrix with positive permanent. It is shown that if the permanent of A can be converted into a determinant by affixing $\pm$ signs to the elements of A then A has at most $(n^2+3n-2)/2$ positive entries. Corollaries of this result are given.

The permanent appears naturally in many combinatorial problems. Since computations with the permanent are difficult, it is of interest to find a simple method for conversion of the permanent into the determinant. Pólya [4] noted that there is no method of uniformly affixing $\pm$ signs to the elements of the matrices of the vector space M_n, $n>2$, of all n-square matrices over the field F of characteristic zero so that the permanent is converted into the determinant. Marcus and Minc [2] generalized this by showing that if $n>2$ then there is no linear transformation $\sigma: M_n \to M_n$ such that per $A=\det \sigma(A)$ for every A in M_n. In this paper, a different improvement of Pólya's result is given. It is shown that if A is an n-square (0, 1)-matrix with positive permanent and there is a way of converting the permanent of A into a determinant by affixing $\pm$ signs to the elements of A then A has at most $(n^2+3n-2)/2$ positive entries.

Let $A=[a_{ij}]$ be an n-square matrix. Let A_{ij} be the $(n-1)$-square submatrix of A that remains after row i and column j are removed, and let s_{ij} denote the sum of the entries in the complement of A_{ij}, i.e.,

$$s_{ij} = \sum_{k=1}^{n} a_{ik} + \sum_{m=1}^{n} a_{mj} - a_{ij}.$$

If there exists an n-square matrix $B=[b_{ij}]$ such that per $A = \det B$ and $b_{ij}=\pm a_{ij}$ for $i, j=1, \cdots, n$, then A is *convertible*. If A contains a $k\times(n-k)$ zero submatrix, for some $1\leqq k\leqq n-1$, then A is *partly decomposable*; otherwise, A is *fully indecomposable*.

If A and B are n-square matrices, let $A\sim B$ denote that there exist

Received by the editors January 30, 1970.

AMS 1969 *subject classifications*. Primary 1520; Secondary 0540, 0525.

Key words and phrases. Permanents, determinants, (0, 1)-matrices, matrices over a field.

[1] This paper was presented at the Auburn University Matrix Theory Conference, June 11, 1970.

permutation matrices P and Q such that $A=PBQ$. Clearly, if A is convertible and $A\sim B$,then B is convertible.

Let $T_n=[t_{ij}]$ be the n-square (0, 1)-matrix with $t_{ij}=0$ if and only if $1\leqq i<j<n$, let $\nu(A)$ denote the number of 1's in the (0, 1)-matrix A, and let $\Omega_n=(n^2+3n-2)/2$. If $A\sim T_n$, then per $A>0$, $\nu(A)=\Omega_n$, and it follows from [1] that A is convertible. In this paper we prove the converse.

We shall use the following three lemmas in our proof of the primary result.

LEMMA 1. *If $A=[a_{ij}]$ is an n-square convertible* (0, 1)-*matrix*, $n\geqq 2$, *and $a_{km}=1$, then A_{km} is convertible.*

PROOF. Let $B=[b_{ij}]$ be an n-square matrix with per $A=\det B$ and $b_{ij}=\pm a_{ij}$. Expanding per A and $\det B$ by row k,

$$\sum_{j=1}^{n} a_{kj} \operatorname{per} A_{kj} = \sum_{j=1}^{n} (-1)^{k+j} b_{kj} \det B_{kj}. \tag{1}$$

Since $b_{ij}=\pm a_{ij}$ and $a_{ij}\geqq 0$,

$$a_{kj} \operatorname{per} A_{kj} \geqq (-1)^{k+j} b_{kj} \det B_{kj}, \qquad j=1,\cdots,n. \tag{2}$$

Since $a_{km}=1=\pm b_{km}$, (1) and (2) imply that per $A_{km}=\pm\det B_{km}$. Hence, A_{km} is convertible.

LEMMA 2. *If $A=[a_{ij}]$ is an n-square* (0, 1)-*matrix*, $n\geqq 3$, *with $a_{jj}=1$ and $\nu(A_{jj})\leqq\Omega_{n-1}$ for $j=1,\cdots,n$, then*

$$\min\{s_{jj}\mid 1\leqq j\leqq n\}\leqq n+2, \tag{3}$$

with equality only if

$$\nu(A)=1+\Omega_n. \tag{4}$$

PROOF. Suppose that

$$s_{kk}=\min\{s_{jj}\mid 1\leqq j\leqq n\}. \tag{5}$$

Since $a_{jj}=1$,

$$ns_{kk}\leqq\sum_{j=1}^{n} s_{jj}=2\nu(A)-n. \tag{6}$$

Since $\nu(A_{kk})\leqq\Omega_{n-1}$,

$$\nu(A)\leqq s_{kk}+\Omega_{n-1}. \tag{7}$$

Combining (5), (6), and (7), we have (3). Suppose that equality holds in (3). Then equality holds in (7). These two equalities imply (4).

LEMMA 3. *If* $A=[a_{ij}]$ *is an n-square* (0, 1)*-matrix*, $n \geqq 5$, *such that*

$$a_{ij} = 1 \Rightarrow s_{ij} \geqq n + 1, \tag{8}$$

$$(a_{ij} = 1,\ s_{ij} = n + 1) \Rightarrow A_{ij} \sim T_{n-1}, \tag{9}$$

$$a_{11} = 1, \qquad s_{11} = n + 1, \tag{10}$$

$$A_{11} = T_{n-1}, \tag{11}$$

then

$$A \sim T_n. \tag{12}$$

PROOF. Suppose that $a_{1n}+a_{n1}=0$. Since $a_{1n}=0$, (11), (8), and (9) imply that $A_{2n}\sim T_{n-1}$. Since $a_{n1}=0$, this implies that

$$a_{1j} = 1, \qquad j = 1, \cdots, n - 1. \tag{13}$$

Similarly, $A_{n,n-1}\sim T_{n-1}$,

$$a_{j1} = 1, \qquad j = 1, \cdots, n - 1. \tag{14}$$

From (13) and (14), $s_{11}=2n-3$. Since $n\geqq 5$, this is a contradiction to (10). Hence, $a_{1n}+a_{n1}\geqq 1$. Combining this with (11) and (8),

$$a_{1j} + a_{j1} \geqq 1, \qquad j = 2, \cdots, n. \tag{15}$$

We consider two cases.

Case (i). Let $a_{1n}+a_{n1}=1$. Suppose that $a_{1n}=1$, $a_{n1}=0$. From (10) and (15),

$$a_{12} = a_{21} = 1 \quad \text{or} \quad a_{13} + a_{31} = 1. \tag{16}$$

Since $a_{n1}=0$ and $n\geqq 5$, (11), (16), and (9) imply that

$$a_{1,n-1} + a_{n-1,1} = 2. \tag{17}$$

From (10), (15), and (17), $a_{12}+a_{21}=1=a_{13}+a_{31}$. Combining this with (11) and (9), $a_{1j}=1, j=1, \cdots, n$. Combining this with (11) and (17), we have (12). If $a_{1n}=0$ and $a_{n1}=1$ a similar argument shows (12).

Case (ii). Let

$$a_{1n} + a_{n1} = 2. \tag{18}$$

Then (10) and (15) imply that

$$a_{1j} + a_{j1} = 1, \qquad j = 2, \cdots, n - 1. \tag{19}$$

If $a_{1,n-1}=1$, we can reduce this case to Case (i) by interchanging row $n-1$ and row n of A. Suppose that $a_{1,n-1}=0$. Then there exists $1\leqq k\leqq n-2$ such that $a_{1k}=1$ and

$$a_{1j} = 0, \qquad j = k + 1, \cdots, n - 1. \tag{20}$$

We shall prove that

$$a_{1j} = 1, \qquad j = 1, \cdots, k. \tag{21}$$

Let r_j be the jth row sum of $A_{k+1,k+1}$. Suppose that $2 \leqq m \leqq k-1$ with $a_{1j}=1, j=1, \cdots, m-1$. It is easy to show that

$$r_1 > m,$$
$$r_j < m, \qquad j = 2, \cdots, m-1,$$
$$r_j > m, \qquad j = m+1, \cdots, n-1.$$

Hence, since $A_{k+1,k+1} \sim T_{n-1}$, we have $r_m = m$. Combining this with (11) and (19), we have $a_{1m}=1$. This implies (21). Combining (11) with (18) through (21), we have (12).

THEOREM. *If A is an n-square convertible* (0, 1)-*matrix with* per $A>0$ *then*

$$\nu(A) \leqq \Omega_n \tag{22}$$

with equality if and only if $A \sim T_n$.

PROOF. Clearly this statement is true for $n=1$, 2, and it is easy to prove for $n=3$, 4. Assume that it is true for all $m<n$, where $n \geqq 5$, and let $A=[a_{ij}]$ be an n-square convertible (0, 1)-matrix with per $A>0$. Suppose that A is partly decomposable. We may assume that

$$A = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix},$$

where A_1 is k-square, $1 \leqq k \leqq n-1$. Since A is a convertible (0, 1)-matrix and (per A_1) (per A_2) = per $A>0$, A_j is a convertible (0, 1)-matrix with per $A_j>0$, $j=1$, 2. Hence, using the inductive assumption,

$$\nu(A) \leqq \Omega_k + k(n-k) + \Omega_{n-k} = \Omega_n - 1.$$

This proves (22), for A partly decomposable.

Now suppose that A is fully indecomposable. We may assume that $a_{11}=1$,

$$s_{11} = \min\{s_{ij} \mid a_{ij} = 1\}. \tag{23}$$

From Minc's characterization of fully indecomposable matrices [**3**],

$$\text{per } A_{ij} > 0, \qquad i, j = 1, \cdots, n. \tag{24}$$

Since per $A_{11}>0$, we may assume that

$$a_{jj} = 1, \qquad j = 1, \cdots, n. \tag{25}$$

According to Lemma 1, A_{jj} is convertible. Hence, from (24) and the inductive assumption,

$$\nu(A_{jj}) \leqq \Omega_{n-1}, \qquad j = 1, \cdots, n. \tag{26}$$

From (23), (25), (26), and Lemma 2,

$$s_{11} \leqq n + 2. \tag{27}$$

Suppose that equality holds in (27). By Lemma 2, we have (4). Hence $\nu(A_{11}) = \Omega_{n-1}$. Hence, from Lemma 1, (24), and the inductive assumption, we have $A_{11} \sim T_{n-1}$. We may assume that

$$A_{11} = T_{n-1}. \tag{28}$$

Since equality holds in (27), (23) and (28) imply that $s_{2n} = s_{n,n-1} = s_{jj} = n+2$, $j = 2, \cdots, n-1$, and therefore that $a_{1j} = a_{j1} = 1$, $j = 2, \cdots, n$. Since $n \geqq 5$, this contradicts (27). Hence

$$s_{11} \leqq n + 1. \tag{29}$$

From (26) and (29), we have (22).

Suppose that equality holds in (22). Then A is fully indecomposable. From (26) and (29), equality must hold in (29) and $\nu(A_{11}) = \Omega_{n-1}$. Hence, by the inductive assumption, $A_{11} \sim T_{n-1}$, and we may assume (11). Since equality holds in (29), we have (8) and (10). It is easy to show (9). Hence by Lemma 3, $A \sim T_n$. The converse follows from [1].

We state three corollaries.

COROLLARY 1. *If A is an n-square convertible* (0, 1)-*matrix*, $n \geqq 5$, *then $\nu(A) \leqq n(n-1)$ with equality only if A has a zero row or a zero column.*

Let M_n be the ring of all n-square matrices over a field F of characteristic zero. If

$$K_n \subset \{1, \cdots, n\} \times \{1, \cdots, n\},$$

let

$$\Gamma(K_n) = \{[a_{ij}] \in M_n \mid a_{ij} = 0 \ \forall \ (i, j) \in K_n\},$$

and let $|K_n|$ be the cardinal number of K_n.

COROLLARY 2. *If every matrix in $\Gamma(K_n)$ is convertible and* per $A \neq 0$ *for some A in $\Gamma(K_n)$, then $|K_n| \geqq (n^2 - 3n + 2)/2$, with equality if and only if there exist permutation matrices P and Q such that*

$$\{PAQ \mid A \in \Gamma(K_n)\} = \{[b_{ij}] \in M_n \mid b_{ij} = 0 \ \forall j > i + 1\}.$$

COROLLARY 3. *If $n \geqq 5$ and every matrix in $\Gamma(K_n)$ is convertible, then*

$$|K_n| \geqq n,$$

with equality only if every matrix in $\Gamma(K_n)$ has a zero row or a zero column.

REFERENCES

1. P. M. Gibson, *An identity between permanents and determinants*, Amer. Math. Monthly **76** (1969), 270–271. MR **39** #2779.

2. M. Marcus and H. Minc, *On the relation between the determinant and the permanent*, Illinois J. Math. **5** (1961), 376–381. MR **26** #5004.

3. H. Minc, *On lower bounds for permanents of* (0, 1) *matrices*, Proc. Amer. Math Soc. **22** (1969), 117–123. MR **39** #6891.

4. G. Pólya, *Aufgabe* 424, Arch. Math. Phys. (3) **20** (1913), 271.

UNIVERSITY OF ALABAMA, HUNTSVILLE, ALABAMA 35807