CONVERSION OF THE PERMANENT INTO THE DETERMINANT

P. M. GIBSON

Abstract. Let \( A \) be an \( n \)-square \((0, 1)\)-matrix with positive permanent. It is shown that if the permanent of \( A \) can be converted into a determinant by affixing \( \pm \) signs to the elements of \( A \) then \( A \) has at most \((n^2+3n-2)/2\) positive entries. Corollaries of this result are given.

The permanent appears naturally in many combinatorial problems. Since computations with the permanent are difficult, it is of interest to find a simple method for conversion of the permanent into the determinant. Pólya [4] noted that there is no method of uniformly affixing \( \pm \) signs to the elements of the matrices of the vector space \( M_n, n>2 \), of all \( n \)-square matrices over the field \( F \) of characteristic zero so that the permanent is converted into the determinant. Marcus and Minc [2] generalized this by showing that if \( n>2 \) then there is no linear transformation \( \sigma: M_n \rightarrow M_n \) such that \( \text{per} \ A = \det \sigma(A) \) for every \( A \) in \( M_n \). In this paper, a different improvement of Pólya's result is given. It is shown that if \( A \) is an \( n \)-square \((0, 1)\)-matrix with positive permanent and there is a way of converting the permanent of \( A \) into a determinant by affixing \( \pm \) signs to the elements of \( A \) then \( A \) has at most \((n^2+3n-2)/2\) positive entries.

Let \( A = [a_{ij}] \) be an \( n \)-square matrix. Let \( A_{ij} \) be the \((n-1)\)-square submatrix of \( A \) that remains after row \( i \) and column \( j \) are removed, and let \( s_{ij} \) denote the sum of the entries in the complement of \( A_{ij} \), i.e.,

\[
 s_{ij} = \sum_{k=1}^{n} a_{ik} + \sum_{m=1}^{n} a_{mj} - a_{ij}.
\]

If there exists an \( n \)-square matrix \( B = [b_{ij}] \) such that per \( A = \det B \) and \( b_{ij} = \pm a_{ij} \) for \( i, j = 1, \ldots, n \), then \( A \) is convertible. If \( A \) contains a \( k \times (n-k) \) zero submatrix, for some \( 1 \leq k \leq n-1 \), then \( A \) is partly decomposable; otherwise, \( A \) is fully indecomposable.

If \( A \) and \( B \) are \( n \)-square matrices, let \( A \sim B \) denote that there exist

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permutation matrices $P$ and $Q$ such that $A = PBQ$. Clearly, if $A$ is convertible and $A \sim B$, then $B$ is convertible.

Let $T_n = [t_{ij}]$ be the $n$-square $(0, 1)$-matrix with $t_{ij} = 0$ if and only if $1 \leq i < j < n$, let $v(A)$ denote the number of 1's in the $(0, 1)$-matrix $A$, and let $\Omega_n = (n^2 + 3n - 2)/2$. If $A \sim T_n$, then per $A > 0$, $v(A) = \Omega_n$, and it follows from [1] that $A$ is convertible. In this paper we prove the converse.

We shall use the following three lemmas in our proof of the primary result.

**Lemma 1.** If $A = [a_{ij}]$ is an $n$-square convertible $(0, 1)$-matrix, $n \geq 2$, and $a_{km} = 1$, then $A_{km}$ is convertible.

**Proof.** Let $B = [b_{ij}]$ be an $n$-square matrix with per $A = \det B$ and $b_{ij} = \pm a_{ij}$. Expanding per $A$ and $\det B$ by row $k$,

$$
\sum_{j=1}^{n} a_{kj} \text{ per } A_{kj} = \sum_{j=1}^{n} (-1)^{k+j} b_{kj} \det B_{kj}.
$$

Since $b_{ij} = \pm a_{ij}$ and $a_{ij} \geq 0$,

$$
 a_{kj} \text{ per } A_{kj} \geq (-1)^{k+j} b_{kj} \det B_{kj}, \quad j = 1, \ldots, n.
$$

Since $a_{km} = 1 = \pm b_{km}$, (1) and (2) imply that per $A_{km} = \pm \det B_{km}$. Hence, $A_{km}$ is convertible.

**Lemma 2.** If $A = [a_{ij}]$ is an $n$-square $(0, 1)$-matrix, $n \geq 3$, with $a_{jj} = 1$ and $v(A_{jj}) \leq \Omega_n$ for $j = 1, \ldots, n$, then

$$
\min\{s_{jj} \mid 1 \leq j \leq n\} \leq n + 2,
$$

with equality only if

$$
v(A) = 1 + \Omega_n.
$$

**Proof.** Suppose that

$$
s_{kk} = \min\{s_{jj} \mid 1 \leq j \leq n\}.
$$

Since $a_{jj} = 1$,

$$
n s_{kk} \leq \sum_{j=1}^{n} s_{jj} = 2v(A) - n.
$$

Since $v(A_{kk}) \leq \Omega_{n-1}$,

$$
v(A) \leq s_{kk} + \Omega_{n-1}.
$$

Combining (5), (6), and (7), we have (3). Suppose that equality holds in (3). Then equality holds in (7). These two equalities imply (4).
Lemma 3. If $A = [a_{ij}]$ is an $n$-square $(0, 1)$-matrix, $n \geq 5$, such that

(8) $a_{ij} = 1 \Rightarrow s_{ij} \geq n + 1$,

(9) $(a_{ij} = 1, s_{ij} = n + 1) \Rightarrow A_{ij} \sim T_{n-1}$,

(10) $a_{11} = 1$, $s_{11} = n + 1$,

(11) $T_{n-1}$,

then

(12) $A \sim T_n$.

Proof. Suppose that $a_{i1} + a_{1j} = 0$. Since $a_{1n} = 0$, (11), (8), and (9) imply that $A_{2n} \sim T_{n-1}$. Since $a_{n1} = 0$, this implies that

(13) $a_{1j} = 1$, $j = 1, \ldots, n - 1$.

Similarly, $A_{n,n-1} \sim T_{n-1}$,

(14) $a_{j1} = 1$, $j = 1, \ldots, n - 1$.

From (13) and (14), $s_{11} = 2n - 3$. Since $n \geq 5$, this is a contradiction to (10). Hence, $a_{i1} + a_{1j} \geq 1$. Combining this with (11) and (8),

(15) $a_{ij} + a_{j1} \geq 1$, $j = 2, \ldots, n$.

We consider two cases.

Case (i). Let $a_{11} + a_{n1} = 1$. Suppose that $a_{1n} = 1$, $a_{n1} = 0$. From (10) and (15),

(16) $a_{12} = a_{21} = 1$ or $a_{12} + a_{21} = 1$.

Since $a_{n1} = 0$ and $n \geq 5$, (11), (16), and (9) imply that

(17) $a_{1,n-1} + a_{n-1,1} = 2$.

From (10), (15), and (17), $a_{12} + a_{21} = 1 = a_{13} + a_{31}$. Combining this with (11) and (9), $a_{12} = 1, j = 1, \ldots, n$. Combining this with (11) and (17), we have (12). If $a_{1n} = 0$ and $a_{n1} = 1$ a similar argument shows (12).

Case (ii). Let

(18) $a_{1n} + a_{n1} = 2$.

Then (10) and (15) imply that

(19) $a_{1j} + a_{j1} = 1$, $j = 2, \ldots, n - 1$.

If $a_{1,n-1} = 1$, we can reduce this case to Case (i) by interchanging row $n-1$ and row $n$ of $A$. Suppose that $a_{1,n-1} = 0$. Then there exists $1 \leq k \leq n-2$ such that $a_{1k} = 1$ and

(20) $a_{1j} = 0$, $j = k + 1, \ldots, n - 1$. 

We shall prove that
\[(21) \quad a_{ij} = 1, \quad j = 1, \cdots, k.\]
Let \(r_j\) be the \(j\)th row sum of \(A_{k+1,k+1}\). Suppose that \(2 \leq m \leq k - 1\) with \(a_{1j} = 1, j = 1, \cdots, m - 1\). It is easy to show that
\[
\begin{align*}
    r_1 &> m, \\
    r_j &< m, \quad j = 2, \cdots, m - 1, \\
    r_j &> m, \quad j = m + 1, \cdots, n - 1.
\end{align*}
\]
Hence, since \(A_{k+1,k+1} \sim T_{n-1}\), we have \(r_m = m\). Combining this with (11) and (19), we have \(a_{1m} = 1\). This implies (21). Combining (11) with (18) through (21), we have (12).

**Theorem.** If \(A\) is an \(n\)-square convertible \((0, 1)\)-matrix with per \(A > 0\) then
\[
(22) \quad v(A) \leq \Omega_n
\]
with equality if and only if \(A \sim T_n\).

**Proof.** Clearly this statement is true for \(n = 1, 2\), and it is easy to prove for \(n = 3, 4\). Assume that it is true for all \(m < n\), where \(n \geq 5\), and let \(A = [a_{ij}]\) be an \(n\)-square convertible \((0, 1)\)-matrix with per \(A > 0\). Suppose that \(A\) is partly decomposable. We may assume that
\[
A = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix},
\]
where \(A_1\) is \(k\)-square, \(1 \leq k \leq n - 1\). Since \(A\) is a convertible \((0, 1)\)-matrix and (per \(A_1\)) (per \(A_2\)) = per \(A > 0\), \(A_j\) is a convertible \((0, 1)\)-matrix with per \(A_j > 0\), \(j = 1, 2\). Hence, using the inductive assumption,
\[
v(A) \leq \Omega_k + k(n - k) + \Omega_{n-k} = \Omega_n - 1.
\]
This proves (22), for \(A\) partly decomposable.

Now suppose that \(A\) is fully indecomposable. We may assume that \(a_{11} = 1\),
\[
(23) \quad s_{11} = \min \{s_{ij} \mid a_{ij} = 1\}.
\]
From Minc's characterization of fully indecomposable matrices [3],
\[
(24) \quad \text{per } A_{ij} > 0, \quad i, j = 1, \cdots, n.
\]
Since per \(A_{11} > 0\), we may assume that
\[
(25) \quad a_{jj} = 1, \quad j = 1, \cdots, n.
\]
According to Lemma 1, \( A_{jj} \) is convertible. Hence, from (24) and the inductive assumption,
\[
(26) \quad \nu(A_{jj}) \leq \Omega_{n-1}, \quad j = 1, \ldots, n.
\]
From (23), (25), (26), and Lemma 2,
\[
(27) \quad s_{11} \leq n + 2.
\]
Suppose that equality holds in (27). By Lemma 2, we have (4). Hence \( \nu(A_{11}) = \Omega_{n-1} \). Hence, from Lemma 1, (24), and the inductive assumption, we have \( A_{11} \sim T_{n-1} \). We may assume that
\[
(28) \quad A_{11} = T_{n-1}.
\]
Since equality holds in (27), (23) and (28) imply that \( s_{2n} = s_{n,n-1} = s_{jj} = n+2, j = 2, \ldots, n-1 \), and therefore that \( a_{1j} = a_{ji} = 1, j = 2, \ldots, n \). Since \( n \geq 5 \), this contradicts (27). Hence
\[
(29) \quad s_{11} \leq n + 1.
\]
From (26) and (29), we have (22).

Suppose that equality holds in (22). Then \( A \) is fully indecomposable. From (26) and (29), equality must hold in (29) and \( \nu(A_{11}) = \Omega_{n-1} \). Hence, by the inductive assumption, \( A_{11} \sim T_{n-1} \), and we may assume (11). Since equality holds in (29), we have (8) and (10). It is easy to show (9). Hence by Lemma 3, \( A \sim T_n \). The converse follows from [1].

We state three corollaries.

**Corollary 1.** If \( A \) is an \( n \)-square convertible \((0, 1)\)-matrix, \( n \geq 5 \), then \( \nu(A) \leq n(n-1) \) with equality only if \( A \) has a zero row or a zero column.

Let \( M_n \) be the ring of all \( n \)-square matrices over a field \( F \) of characteristic zero. If
\[
K_n \subset \{1, \ldots, n\} \times \{1, \ldots, n\},
\]
let
\[
\Gamma(K_n) = \{ [a_{ij}] \in M_n \mid a_{ij} = 0 \ \forall \ (i, j) \in K_n \},
\]
and let \( |K_n| \) be the cardinal number of \( K_n \).

**Corollary 2.** If every matrix in \( \Gamma(K_n) \) is convertible and \( \text{per} \ A \neq 0 \) for some \( A \) in \( \Gamma(K_n) \), then \( |K_n| \geq \frac{n^2 - 3n + 2}{2} \), with equality if and only if there exist permutation matrices \( P \) and \( Q \) such that

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\[
\{ P A Q \mid A \in \Gamma(K_n) \} = \{ [b_{ij}] \in M_n \mid b_{ij} = 0 \ \forall \ j > i + 1 \}.
\]

**Corollary 3.** If \( n \geq 5 \) and every matrix in \( \Gamma(K_n) \) is convertible, then

\[
| K_n | \geq n,
\]

with equality only if every matrix in \( \Gamma(K_n) \) has a zero row or a zero column.

**References**


University of Alabama, Huntsville, Alabama 35807