PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 27, Number 3, March 1971

# CHARACTERIZATIONS OF THE GENERALIZED CONVEX KERNEL

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ABSTRACT. It is well known that the convex kernel $K$ of a set $S$ is the intersection of all maximal convex subsets of $S$. In this paper it is shown that the $n$th order kernel of a compact, simply-connected set $S$ in the plane is an $L_n$ set and is, in fact, the intersection of all maximal $L_n$ subsets of $S$. Furthermore, it is shown that one does not have to intersect the family of all the maximal $L_n$ subsets to obtain the $n$th order kernel, but that any subfamily thereof which covers the set is sufficient.

1. **Preliminaries.** Throughout this paper, all sets will be in $E_2$. If $B$ is a set, then $\overline{B}$ will denote its closure, bd $B$ its boundary, and $B^c$ its complement. If $x$ and $y$ are points, then $P_n(x, y)$ will denote a polygonal $n$-path joining $x$ to $y$.

Let $A$ be a set and let $x$ be in $A$. Then $K(n, x, A)$ will denote the $n$th order kernel of $x$ in $A$. The $n$th order kernel of $A$ will be denoted by $K(n, A)$. For precise definitions of $K(n, x, A)$ and $K(n, A)$, see [2].

DEFINITION. A compact set $S$ is said to be simply-connected if and only if $S^c$ is connected.

Hereafter, $S$ will denote a compact, simply-connected set. It is important to remember that if $J$ is a closed Jordan curve in $S$, then the interior of the Jordan curve $J$ is contained in $S$.

DEFINITION. Suppose that $p, q \in S$ and $C(p, q)$ is a polygonal path from $p$ to $q$ in $S$. Then $C(p, q)$ is called a *minimal* 1-*path* if $C(p, q)$ is the segment $[p, q]$. Let $k>1$, then $C(p, q)$ is called a *minimal k-path* if $C(p, q)$ is a $k$-path of minimal length joining $p$ to $q$ in $S$ and $p \notin K(k-1, q, S)$.

Several results in a previous paper by this author [2] will be stated for later use.

THEOREM 1.1. *Let $A$ be a set and let $B$ be an $L_n$ subset of $A$. Then $B$ is contained in a maximal $L_n$ subset of $A$.*

THEOREM 1.2. *Suppose $p \in K(m, q, S)$ for some $m$. Then there exists a minimal $k$-path from $p$ to $q$ in $S$ for some $k$ such that $1 \leq k \leq m$.*

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Presented to the Society, August 29, 1969 under the title *A characterization of the generalized convex kernel*; received by the editors December 7, 1969 and, in revised form, April 15, 1970.

*AMS* 1970 *subject classifications*. Primary 52-XX, 52A10.

*Key words and phrases*. Convex kernel, generalized convex kernel, $L_n$ sets.

THEOREM 1.3. *Suppose that* $p, q \in K(n, x, S)$. *Let* $C_k(p, q)$ *be a minimal k-path from* $p$ *to* $q$ *in* $S$, *then* $C_k(p, q) \subset K(n, x, S)$.

THEOREM 1.4. *Let* $\mathfrak{L}_n = \{L_\alpha | \alpha \in \Delta_n\}$ *be the set of all maximal* $L_n$ *subsets of* $S$. *Then each* $L_\alpha$ *is compact and simply-connected. Furthermore,* $\bigcap \mathfrak{L}_n$ *is a compact, simply-connected,* $L_n$ *set.*

## 2. **Characterizations of** $K(n, S)$.

THEOREM 2.1. *Let* $A$ *be a compact,* $L_n$ *subset of* $S$. *Suppose* $x \in S$ *is such that* $A \subset K(n, x, S)$. *Then* $A \cup \{x\}$ *is contained in an* $L_n$ *subset of* $S$.

PROOF. Let $a \in A$. Since $x \in K(n, a, S)$, it follows from Theorem 1.2 that there exists a minimal $k(a)$-path $C_{k(a)}(a, x)$ joining $a$ to $x$ in $S$, where $k(a) \leq n$. Let $G = \bigcup \{C_{k(a)}(a, x) | a \in A\}$ and let $D$ be the smallest compact, simply-connected set in $S$ which contains $G$.

Suppose that $p, q \in G$. Then there exist $a_1, a_2 \in A$ such that $p \in C_{k(a_1)}(a_1, x)$ and $q \in C_{k(a_2)}(a_2, x)$. It is clear that $x \in K(n, a_1, D)$ and $a_2 \in K(n, a_1, D)$. Since $C_{k(a_2)}(a_2, x) \subset D$, it follows from Theorem 1.3 that $C_{k(a_2)}(a_2, x) \subset K(n, a_1, D)$. In particular, $q \in K(n, a_1, D)$ and thus $a_1 \in K(n, q, D)$. Now since $x \in K(n, q, D)$ and $C_{k(a_1)}(a_1, x)$ is also a minimal $k(a_1)$-path in $D$, it follows again by Theorem 1.3 that $C_{k(a_1)}(a_1, x) \subset K(n, q, D)$. In particular, it is true that $p \in K(n, q, D)$.

Now suppose that $p, q \in \text{bd } D$. It is clear that bd $D \subset \overline{G}$. Since $p, q \in \overline{G}$, there exist sequences $\{p_i\}$ and $\{q_i\}$ in $G$ such that $\{p_i\} \to p$ and $\{q_i\} \to q$.

Let $i$ and $j$ be arbitrary, then from the preceding it follows that $p_i \in K(n, q_j, D)$ since $p_i, q_j \in G$. Now $i$ arbitrary implies that $\{p_i\} \subset K(n, q_j, D)$. It has been shown by Bruckner and Bruckner [1], that $K(n, q_j, D)$ is compact. Thus, it follows that $p \in K(n, q_j, D)$. Hence, $q_j \in K(n, p, D)$ where $j$ is arbitrary and thus $\{q_i\} \subset K(n, p, D)$. As before, $K(n, p, D)$ is compact and hence $q \in K(n, p, D)$. Since $p, q \in \text{bd } D$ were arbitrary, it follows by another result of Bruckner and Bruckner [1] that $D$ is an $L_n$ set. It is clear that $A \cup \{x\} \subset D \subset S$. This completes the proof.

THEOREM 2.2. *Let* $\mathfrak{L}_n = \{L_\alpha | \alpha \in \Delta_n\}$ *be the set of all maximal* $L_n$ *subsets of* $S$. *Then* $K(n, S) = \bigcap \mathfrak{L}_n$.

PROOF. Clearly, $\bigcap \mathfrak{L}_n \subset K(n, S)$.

Suppose that $x$ is in $K(n, S)$ but not in $L_\alpha$, for some $\alpha \in \Delta_n$. Since $L_\alpha \subset K(n, x, S)$, Theorem 2.1 implies that $L_\alpha$ and $x$ are both contained in an $L_n$ subset of $S$, contradicting the maximality of $L_\alpha$. Hence, $K(n, S) \subset \bigcap \mathfrak{L}_n$.

Combining the above, the desired result is obtained.

The same technique can be used to prove the following result:

THEOREM 2.3. *Let* $\mathcal{L}'_n \subset \mathcal{L}_n$ *be such that* $\bigcup \mathcal{L}'_n = S$, *then* $\bigcap \mathcal{L}'_n = K(n, S)$.

THEOREM 2.4. *The set* $K(n, S)$ *is an* $L_n$ *set.*

PROOF. Combine Theorems 1.4 and 2.2.

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