CHARACTERIZATIONS OF THE GENERALIZED CONVEX KERNEL

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Abstract. It is well known that the convex kernel $K$ of a set $S$ is the intersection of all maximal convex subsets of $S$. In this paper it is shown that the $n$th order kernel of a compact, simply-connected set $S$ in the plane is an $L_n$ set and is, in fact, the intersection of all maximal $L_n$ subsets of $S$. Furthermore, it is shown that one does not have to intersect the family of all the maximal $L_n$ subsets to obtain the $n$th order kernel, but that any subfamily thereof which covers the set is sufficient.

1. Preliminaries. Throughout this paper, all sets will be in $E_2$. If $B$ is a set, then $\overline{B}$ will denote its closure, $\text{bd } B$ its boundary, and $B^c$ its complement. If $x$ and $y$ are points, then $P_n(x, y)$ will denote a polygonal $n$-path joining $x$ to $y$.

Let $A$ be a set and let $x$ be in $A$. Then $K(n, x, A)$ will denote the $n$th order kernel of $x$ in $A$. The $n$th order kernel of $A$ will be denoted by $K(n, A)$. For precise definitions of $K(n, x, A)$ and $K(n, A)$, see [2].

Definition. A compact set $S$ is said to be simply-connected if and only if $S^c$ is connected.

Hereafter, $S$ will denote a compact, simply-connected set. It is important to remember that if $J$ is a closed Jordan curve in $S$, then the interior of the Jordan curve $J$ is contained in $S$.

Definition. Suppose that $p, q \in S$ and $C(p, q)$ is a polygonal path from $p$ to $q$ in $S$. Then $C(p, q)$ is called a minimal $1$-path if $C(p, q)$ is the segment $[p, q]$. Let $k > 1$, then $C(p, q)$ is called a minimal $k$-path if $C(p, q)$ is a $k$-path of minimal length joining $p$ to $q$ in $S$ and $p \in K(k-1, q, S)$.

Several results in a previous paper by this author [2] will be stated for later use.

Theorem 1.1. Let $A$ be a set and let $B$ be an $L_n$ subset of $A$. Then $B$ is contained in a maximal $L_n$ subset of $A$.

Theorem 1.2. Suppose $p \in K(m, q, S)$ for some $m$. Then there exists a minimal $k$-path from $p$ to $q$ in $S$ for some $k$ such that $1 \leq k \leq m$.

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Theorem 1.3. Suppose that \( p, q \in K(n, x, S) \). Let \( C_k(p, q) \) be a minimal \( k \)-path from \( p \) to \( q \) in \( S \), then \( C_k(p, q) \subset K(n, x, S) \).

Theorem 1.4. Let \( L_n = \{ L_\alpha | \alpha \in \Delta_n \} \) be the set of all maximal \( L_n \) subsets of \( S \). Then each \( L_\alpha \) is compact and simply-connected. Furthermore, \( \bigcap L_n \) is a compact, simply-connected, \( L_n \) set.

2. Characterizations of \( K(n, S) \).

Theorem 2.1. Let \( A \) be a compact, \( L_n \) subset of \( S \). Suppose \( x \in S \) is such that \( A \in K(n, x, S) \). Then \( A \cup \{ x \} \) is contained in an \( L_n \) subset of \( S \).

Proof. Let \( a \in A \). Since \( x \in K(n, a, S) \), it follows from Theorem 1.2 that there exists a minimal \( k(a) \)-path \( C_{k(a)}(a, x) \) joining \( a \) to \( x \) in \( S \), where \( k(a) \leq n \). Let \( G = \bigcup \{ C_{k(a)}(a, x) | a \in A \} \) and let \( D \) be the smallest compact, simply-connected set in \( S \) which contains \( G \).

Suppose that \( p, q \in G \). Then there exist \( a_0, a_1 \in A \) such that \( p \in C_{k(a_0)}(a_0, x) \) and \( q \in C_{k(a_0)}(a_1, x) \). It is clear that \( x \in K(n, a_1, D) \) and \( a_2 \in K(n, a_1, D) \). Since \( C_{k(a_2)}(a_2, x) \subset D \), it follows from Theorem 1.3 that \( C_{k(a_2)}(a_2, x) \subset K(n, a_1, D) \). In particular, \( q \in K(n, a_1, D) \) and thus \( a_1 \in K(n, q, D) \). Now since \( x \in K(n, q, D) \) and \( C_{k(a_1)}(a_1, x) \) is also a minimal \( k(a_1) \)-path in \( D \), it follows again by Theorem 1.3 that \( C_{k(a_1)}(a_1, x) \subset K(n, q, D) \). In particular, it is true that \( p \in K(n, q, D) \).

Now suppose that \( p, q \in E \) \( D \). It is clear that \( \partial D \subset G \). Since \( p, q \in G \), there exist sequences \( \{ p_i \} \) and \( \{ q_i \} \) in \( G \) such that \( \{ p_i \} \to p \) and \( \{ q_i \} \to q \).

Let \( i \) and \( j \) be arbitrary, then from the preceding it follows that \( p_i \in K(n, q_j, D) \) since \( p_i, q_j \in G \). Now \( i \) arbitrary implies that \( \{ p_i \} \subset K(n, q_j, D) \). It has been shown by Bruckner and Bruckner [1], that \( K(n, q_j, D) \) is compact. Thus, it follows that \( p \in K(n, q_j, D) \). Hence, \( q_j \in K(n, p, D) \) where \( j \) is arbitrary and thus \( \{ q_i \} \subset K(n, p, D) \). As before, \( K(n, p, D) \) is compact and hence \( q \in K(n, p, D) \). Since \( p, q \in \partial D \) were arbitrary, it follows by another result of Bruckner and Bruckner [1] that \( D \) is an \( L_n \) set. It is clear that \( A \cup \{ x \} \subset D \subset S \).

This completes the proof.

Theorem 2.2. Let \( L_n = \{ L_\alpha | \alpha \in \Delta_n \} \) be the set of all maximal \( L_n \) subsets of \( S \). Then \( K(n, S) = \bigcap L_n \).

Proof. Clearly, \( \bigcap L_n \subset K(n, S) \).

Suppose that \( x \) is in \( K(n, S) \) but not in \( L_\alpha \), for some \( \alpha \in \Delta_n \). Since \( L_\alpha \subset K(n, x, S) \), Theorem 2.1 implies that \( L_\alpha \) and \( x \) are both contained in an \( L_n \) subset of \( S \), contradicting the maximality of \( L_\alpha \). Hence, \( K(n, S) \subset \bigcap L_n \).
Combining the above, the desired result is obtained. The same technique can be used to prove the following result:

**Theorem 2.3.** Let $\mathcal{L}'_n \subseteq \mathcal{L}_n$ be such that $\bigcup \mathcal{L}'_n = S$, then $\bigcap \mathcal{L}'_n = K(n, S)$.

**Theorem 2.4.** The set $K(n, S)$ is an $L_n$ set.

**Proof.** Combine Theorems 1.4 and 2.2.

**References**


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