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ABSTRACT. Let $f(z)$ be a normalized function, analytic and univalent in the open unit disc. It is shown that if $g(z)=\int_0^z (f(t)/t)^\alpha dt$, then g is univalent in the open unit disc if α is a complex number satisfying $0 \leqq |\alpha| \leqq (\sqrt{2}-1)/4$.

1. **Introduction.** Let S be the class of functions $f(z)=z+a_2z^2+\cdots$ analytic and univalent in the open unit disc E. Define

$$g(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt.$$

The author proved in [1] that if $f \in S$, then $g \in S$ if α is a complex number satisfying $0 \leqq |\alpha| \leqq (\sqrt{5}-2)/4$. This bound has been improved by M. Nunokawa [5] to $0 \leqq |\alpha| \leqq \alpha_0$, where

$$\frac{(18425)^{1/2} - 75}{800} < \alpha_0 < \frac{(24841)^{1/2} - 125}{384}.$$

These results were obtained by a method of P. L. Duren, H. S. Shapiro, and A. L. Shields [2] that utilizes a remarkable univalence criterion of Z. Nehari involving the Schwarzian derivative. The purpose of this paper is to improve the value of α_0 to $(\sqrt{2}-1)/4$.

2. We will require some lemmas. The first is Nehari's result and may be found in [3]. Lemma 2 is due to the author [1, p. 209].

LEMMA 1. *Suppose that $w=f(z)$ is analytic in E and $|\{w, z\}| \leqq 2/(1-r^2)^2$ for all $z \in E$, $|z|=r$, where $\{w, z\}=(w''/w')'-\frac{1}{2}(w''/w')^2$ is the Schwarzian derivative. Then $f(z)$ is univalent in E.*

LEMMA 2. *Suppose $f \in S$ and $g(z)=\int_0^z (f(t)/t)^\alpha dt$, then*

$$|g''(z)/g'(z)| \leqq 2|\alpha|/r(1-r) \tag{1}$$

for $0<r=|z|<1$, and

$$|g''(z)/g'(z)| \leqq 8|\alpha|/(1-r^2) \tag{2}$$

for $0 \leqq r < 1$.

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LEMMA 3. *Suppose $h(z)$ is analytic in E and $|h(z)| \leqq 1/r(1-r)$, $0<r=|z|<1$. Then*

$$(3) \qquad |h'(z)| \leqq 27/2\{1-9r^2+(1+3r^2)^{3/2}\} \equiv B(r),$$

for $0 \leqq r<1$.

PROOF. By the Cauchy integral formula

$$h'(z) = \frac{1}{2\pi i}\int_C \frac{h(t)}{(t-z)^2}\,dt,$$

where C is the circle $|t|=R$, $|z|=r<R<1$. Then, if $t=Re^{i\phi}$,

$$(4) \qquad |h'(z)| \leqq \frac{1}{2\pi(1-R)}\int_0^{2\pi}\frac{d\phi}{|z-Re^{i\phi}|^2} = \frac{1}{(1-R)(R^2-r^2)},$$

for the integral in (4) is a Poisson integral. But this expression takes its minimum in $(r, 1)$ at $R=\frac{1}{3}+\frac{1}{3}\sqrt{(1+3r^2)}$. Substitution gives the result.

THEOREM. *Suppose $f \in S$ and $g(z)=\int_0^z (f(t)/t)^\alpha dt$. Then $g \in S$ for $0 \leqq |\alpha| \leqq (\sqrt{2}-1)/4$.*

PROOF. From (1) and Lemma 3 it follows that

$$(5) \qquad |(g''(z)/g'(z))'| \leqq 2|\alpha| B(r).$$

From Lemma 1, (2), and (5) we have

$$\begin{aligned} |\{g, z\}| &\leqq \tfrac{1}{2}|(g''/g')|^2 + |(g''/g')'| \\ &\leqq \frac{32|\alpha|^2}{(1-r^2)^2} + 2|\alpha| B(r) \\ &= \frac{2}{(1-r^2)^2}\{16|\alpha|^2 + (1-r^2)^2|\alpha| B(r)\}. \end{aligned}$$

Define $M(r)=(1-r^2)^2B(r)$. A calculation shows that $(d/dr)M(\sqrt{r})>0$, $r \in [0, 1)$, and so

$$M(r) < \lim_{r\to 1} M(r) = 8, \qquad 0 \leqq r < 1.$$

Hence $|\{g, z\}| \leqq 2/(1-r^2)^2$ if

$$16|\alpha|^2 + 8|\alpha| - 1 \leqq 0,$$

which is true if $0 \leqq |\alpha| \leqq (\sqrt{2}-1)/4$, and the theorem follows from Lemma 1.

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