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A COMPARISON THEOREM FOR OPERATORS WITH COMPACT RESOLVENT

REESE T. PROSSER

ABSTRACT. The asymptotic behavior of the eigenvalue sequence of a semibounded operator with compact resolvent is stable under relatively bounded perturbations.

It is sometimes useful to have available results providing a means of comparing the "size" of a given operator with that of another [5]. This is particularly true when their difference is "small," so that one of them can be regarded as a perturbation of the other. We present here an elementary result of this nature, valid for operators with compact resolvent acting on a Hilbert space, which derives from the work of Erhard Heinz [4]. We obtain from this result that the asymptotic behavior of the eigenvalue sequence of such an operator is stable under relatively bounded perturbations, thus providing a modest improvement on a recent result of Richard Beals [1].

Let H be a Hilbert space and A a closed nonnegative operator with domain $\operatorname{dom}(A)$ dense in H. Denote by $A^{1/2}$ the unique closed nonnegative square root of A.

Let B be another such operator, and suppose

(a) $\operatorname{dom}(A^{1/2}) \supset \operatorname{dom}(B^{1/2})$, and

(b) $\|A^{1/2}x\| \leqq \|B^{1/2}x\|$ for all $x \in \operatorname{dom}(B^{1/2})$.

Then we say B *majorizes* A, and write $A \leqq B$. If B is bounded and majorizes A, then clearly A is bounded, and $(Ax, x) \leqq (Bx, x)$ for all $x \in H$. Moreover, if $I \leqq B$, then B is invertible, and $I \geqq B^{-1}$.

For such operators we have the following sequence of results:

LEMMA 1. *If* $\operatorname{dom}(A) \supset \operatorname{dom}(B)$, *then for some positive constant* k *we have* $(I+A)^2 \leqq k^2(I+B)^2$.

PROOF. Define on $\operatorname{dom}(A)$ and $\operatorname{dom}(B)$ the norms $\|x\|_A = \|(I+A)x\|$ and $\|x\|_B = \|(I+B)x\|$, respectively, and note that because A and B are closed, $\operatorname{dom}(A)$ and $\operatorname{dom}(B)$ become Hilbert spaces under these norms. Moreover, the injection J: $\operatorname{dom}(B) \to \operatorname{dom}(A)$ is evidently closed under these norms. By the Closed Graph Theorem, J is bounded, and $\|x\|_A = \|Jx\|_A \leqq \|J\| \, \|x\|_B$ for all $x \in \operatorname{dom}(B)$, as required [3].

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LEMMA 2. *If* $(I+A)^2 \leqq k^2(I+B)^2$, *then* $(I+A) \leqq k(I+B)$.

LEMMA 3. *If* $(I+A) \leqq k(I+B)$, *then* $(I+A)^{-1} \geqq k^{-1}(I+B)^{-1}$.

The proofs of these lemmas are found in the work of Erhard Heinz [4], who provided a general framework for the study of such questions.

Now suppose that $(I+A)^{-1}$ is *compact*. Then we know that A has a pure point spectrum, and we can arrange the eigenvalues in order of increasing magnitude: $0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n \uparrow \infty$. Moreover, if $(I+A)^{-1} \geqq k(I+B)^{-1}$, then $(I+B)^{-1}$ is also compact, and we can arrange the eigenvalues b_n of B similarly.

LEMMA 4. *If* $(I+A)^{-1}$ *is compact, and* $(I+A)^{-1} \geqq k^{-1}(I+B)^{-1}$ *then* $(1+a_n)^{-1} \geqq k^{-1}(1+b_n)^{-1}$ *for all* n.

PROOF. This follows from the variational definition of the eigenvalues $(1+a_n)^{-1}$ and $(1+b_n)^{-1}$ of $(I+A)^{-1}$ and $(I+B)^{-1}$, respectively [3].

Combining these results, we obtain

THEOREM 5. *If* $(I+A)^{-1}$ *is compact, and* $\operatorname{dom}(A) \supset \operatorname{dom}(B)$, *then* $(I+B)^{-1}$ *is compact, and for some positive constant* k *we have* $(1+a_n) \leqq k(1+b_n)$ *for all* n.

Now write $V = A - B$, and suppose V is *bounded relative to* B: $\|Vx\| \leqq c\|(I+B)x\|$ for some $c>0$ and all $x \in \operatorname{dom}(B)$. Then we have

$$\|(I+A)x\| \leqq \|(I+B)x\| + \|Vx\| \leqq (1+c)\|(I+B)x\|.$$

Moreover, if $c<1$, then

$$\|Vx\| \leqq c\|(I+A-V)x\| \leqq c\|(I+A)x\| + c\|Vx\|,$$

so $(1-c)\|Vx\| \leqq c\|(I+A)x\|$ and $\|Vx\| \leqq c(1-c)^{-1}\|(I+A)x\|$. Hence,

$$\|(I+B)x\| \leqq \|(I+A)x\| + \|Vx\| \leqq (1-c)^{-1}\|(I+A)x\|.$$

Under these circumstances we have

$$(1-c)\|(I+B)x\| \leqq \|(I+A)x\| \leqq (1+c)\|(I+B)x\|.$$

COROLLARY. *If in addition* $\|(A-B)x\| \leqq c\|(I+B)x\|$ *for all* $x \in \operatorname{dom}(B)$ *and some constant* $0<c<1$, *then we have* $\operatorname{dom}(A) = \operatorname{dom}(B)$, *and*

$$(1-c)(1+a_n) \leqq (1+b_n) \leqq (1+c)(1+a_n) \qquad \textit{for all } n.$$

These results extend easily to semibounded operators by replacing

A by $A+a\ I$ for large $a>0$, and to general operators by replacing A by $(AA^*)^{1/2}$ [5].

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DARTMOUTH COLLEGE, HANOVER, NEW HAMPSHIRE 03755