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ON THE VANISHING OF Ext

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ABSTRACT. In this paper we exhibit certain modules A over a commutative noetherian local ring $(R, \mathfrak{M})$ which test projective dimension of finitely generated modules in the following sense: if $\operatorname{Ext}^j(M, A)=0$ for all $j \geqq i$, then pd $M<i$.

We also show that the module $\mathfrak{M}$ tests in a stronger way: if $\operatorname{Ext}^i(M, \mathfrak{M})=0$, then pd $M<i$.

In conclusion we show that if $(R, \mathfrak{M})$ is artin, then R is self-injective if and only if $\operatorname{Ext}^1(R/\mathfrak{M}^n, R)=0$, where the index of nilpotence of $\mathfrak{M}$ is $n+1$.

Let $(R, \mathfrak{M})$ be a commutative, noetherian, local ring with maximal ideal $\mathfrak{M}$. It is well known that for any finitely generated R-module M, if $\operatorname{Ext}^i(M, R/\mathfrak{M})=0$, then the projective dimension of M(pd M) is less than i. We shall say that $R/\mathfrak{M}$ is a strong test module. In this paper we exhibit another strong test module, namely $\mathfrak{M}$.

We define a weak test module as a module A such that for any finitely generated module M, if $\operatorname{Ext}^j(M, A)=0$ for all $j \geqq 1$, then pd $M<i$. The following theorem is proved: Let $e_0=1$ and, for $i \geqq 1$, let $e_i=[\mathfrak{M}^i/\mathfrak{M}^{i+1}: R/\mathfrak{M}]$ (vector space dimension). If $e_n > \sum_{i=0}^{n-1} e_i$ then $R/\mathfrak{M}^{n+1}$ is a weak test module. As a corollary we show that if R is regular local of dimension k and $S=R/\mathfrak{M}^{n+1}$, $n+1 \leqq k$, then S is a weak test module as an S-module.

We conclude with the observation that if R is artin then R is self-injective if and only if $\operatorname{Ext}^1(R/\mathfrak{M}^n, R)=0$ where the index of nilpotence of $\mathfrak{M}$ is $n+1$.

Notation and conventions. Throughout this paper $(R, \mathfrak{M})$ will denote a commutative, noetherian, local ring with maximal ideal $\mathfrak{M}$ and with a unit element. All modules are unital and finitely generated. By pd M we mean the projective dimension of M. For an $R/\mathfrak{M}$-module E, $[E: R/\mathfrak{M}]$ will denote the vector space dimension of E. We shall denote the direct sum of q copies of M by $\bigoplus \sum_q M$.

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1. Strong test modules. We begin with the following

DEFINITION. An R-module A is a *strong test module* if for all R-modules M and for all $i>0$, $\mathrm{Ext}^i(M, A)=0$ implies that $\mathrm{pd}\, M<i$.

By taking a minimal resolution of M it is easy to see that $\mathrm{Ext}^i(M, R/\mathfrak{M})=X_i/\mathfrak{M}X_i$, where X_i is the ith module in the resolution. Nakayama's Lemma shows that $\mathrm{Ext}^i(M, R/\mathfrak{M})=0$ if and only if $X_i=0$. Since the resolution was minimal, this happens if and only if $\mathrm{pd}\, M<i$. Hence $R/\mathfrak{M}$ is a strong test module.

LEMMA 1.1. *A is a strong test module$\Leftrightarrow$for all modules M, $\mathrm{Ext}^1(M, A)=0$ implies that M is projective.*

PROOF. ($\Rightarrow$) Obvious.

($\Leftarrow$) By induction on i. We have the case $i=1$ by assumption. Let $0\to K\to F\to M\to 0$ be exact with F free. Then $\mathrm{Ext}^i(K, A)\cong\mathrm{Ext}^{i+1}(M, A)$. Supose the theorem is true for i and $\mathrm{Ext}^{i+1}(M, A)=0$. Then $\mathrm{Ext}^i(K, A)=0$, so $\mathrm{pd}\, K<i$, and therefore $\mathrm{pd}\, M<i+1$. ■

LEMMA 1.2. *If $M\neq 0$ and $\mathrm{Ext}^1(M, \mathfrak{M})=0$ then R is a direct summand of M.*

PROOF. Apply the functor $\mathrm{Hom}(M, \ )$ to the exact sequence

$$0\to\mathfrak{M}\to R\to R/\mathfrak{M}\to 0.$$

Then $0\to\mathrm{Hom}(M, \mathfrak{M})\to\mathrm{Hom}(M, R)\to\mathrm{Hom}(M, R/\mathfrak{M})\to 0$ is exact. Let $f\in\mathrm{Hom}(M, R)$. If $f(M)=R$ then the sequence $0\to\mathrm{Ker}\, f\to M\to R\to 0$ splits, since R is projective, and we have the desired result. So assume that for all $f\in\mathrm{Hom}(M, R)$, $f(M)\neq R$. Then $f(M)\subset\mathfrak{M}$. Hence the map $\mathrm{Hom}(M, R)\to\mathrm{Hom}(M, R/\mathfrak{M})$ is the zero map. But it is also onto, so $\mathrm{Hom}(M, R/\mathfrak{M})=0$. From the epimorphism $M\to M/\mathfrak{M}M\to 0$ we have the injection $0\to\mathrm{Hom}(M/\mathfrak{M}M, R/\mathfrak{M})\to\mathrm{Hom}(M, R/\mathfrak{M})$. Thus $\mathrm{Hom}(M/\mathfrak{M}M, R/\mathfrak{M})=0$, which, since $M/\mathfrak{M}M$ is a vector space over $R/\mathfrak{M}$, implies that $M/\mathfrak{M}M=0$. Nakayama's Lemma says that $M=0$, which is a contradiction. ■

THEOREM 1.3. *If $M\neq 0$ and $\mathrm{Ext}^1(M, \mathfrak{M})=0$ then M is R-free. Consequently $\mathfrak{M}$ is a strong test module.*

PROOF. By Lemma 1.2, $M\cong M_1\oplus R$. Since Ext commutes with direct sums, $\mathrm{Ext}^1(M_1, \mathfrak{M})=0$. By the same lemma, if $M_1\neq 0$ then $M_1\cong M_2\oplus R$. Thus $M\cong M_2\oplus R\oplus R$. We may keep repeating this process, except that M is finitely generated, so at some point $M_{i+1}=0$ and $M_i=R$. Hence $M\cong\bigoplus\sum R$ and so it is R-free.

The second statement follows from Lemma 1.1. ■

PROPOSITION 1.4. *If A is a strong test module and $x \in \mathfrak{M}$ is A-regular (i.e. x is not a zero divisor on A), then A/xA is a strong test module.*

PROOF. Suppose $\mathrm{Ext}^1(M, A/xA) = 0$ for some module M. From the exact sequence $0 \to A \xrightarrow{x} A \to A/xA \to 0$ we have that $\mathrm{Ext}^1(M, A) \xrightarrow{x} \mathrm{Ext}^1(M, A) \to 0$ is exact. But $\mathrm{Ext}^1(M, A)$ is finitely generated since M and A are, so by Nakayama's Lemma $\mathrm{Ext}^1(M, A) = 0$. Hence M is R-free. ∎

The next proposition shows that the class of strong test modules is rather limited.

PROPOSITION 1.5. *If A is a strong test module then* $\mathrm{depth}_R\, A \leqq 1$.

PROOF. $\mathrm{depth}_R\, A$ = the length of the longest A-regular sequence contained in $\mathfrak{M}$ = the least integer $i \geqq 0$ such that $\mathrm{Ext}^i(R/\mathfrak{M}, A) \neq 0$. Thus if $\mathrm{depth}_R A > 1$ then $\mathrm{Ext}^1(R/\mathfrak{M}, A) = 0$ and hence $R/\mathfrak{M}$ is R-free. This means that $\mathfrak{M} = 0$, which implies that $\mathrm{depth}_R\, A = 0$, contrary to our assumption. So $\mathrm{depth}_R\, A \leqq 1$. ∎

2. Weak test modules.

DEFINITION. A is a *weak test module* if for all modules M and for all $i > 0$, if $\mathrm{Ext}^j(M, A) = 0$ for all $j \geqq 1$ then M is R-free.

The following lemma is the analogue of Lemma 1.1; its proof is nearly identical to that of Lemma 1.1 and we omit it.

LEMMA 2.1. *A is a weak test module $\Leftrightarrow$ for all modules M, if $\mathrm{Ext}^j(M, A) = 0$ for all $j \geqq 1$ then M is R-free.*

NOTATION. Let $e_0 = 1$ and, for $i > 0$, let $e_i = [\mathfrak{M}^i/\mathfrak{M}^{i+1} : R/\mathfrak{M}]$.

THEOREM 2.2. *Suppose $e_n > \sum_{i=0}^{n-i} e_i$, for some $n \geqq 1$. If*

$$\mathrm{Ext}^i(M, R/\mathfrak{M}^{n+1}) = 0 \quad \textit{for all } i > 0$$

then M is R-free. In other words, $R/\mathfrak{M}^{n+1}$ is a weak test module.

PROOF. The following sequences are all exact:

$$(1) \qquad 0 \to \mathfrak{M}^n/\mathfrak{M}^{n+1} \to R/\mathfrak{M}^{n+1} \to R/\mathfrak{M}^n \to 0,$$

$$(2) \qquad 0 \to \mathfrak{M}^{n-1}/\mathfrak{M}^n \to R/\mathfrak{M}^n \to R/\mathfrak{M}^{n-1} \to 0,$$

$$\vdots$$

$$(n) \qquad 0 \to \mathfrak{M}/\mathfrak{M}^2 \to R/\mathfrak{M}^2 \to R/\mathfrak{M} \to 0.$$

If we apply the functor $\mathrm{Hom}(M, \;\;)$ to each of these sequences, we obtain long exact sequences from which we extract the following, for all $i > 0$:

$$(1') \qquad \mathrm{Ext}^i(M, R/\mathfrak{M}^n) \cong \mathrm{Ext}^{i+1}(M, \mathfrak{M}^n/\mathfrak{M}^{n+1}),$$

$$(2') \qquad \mathrm{Ext}^i(M, \mathfrak{M}^{n-1}/\mathfrak{M}^n) \to \mathrm{Ext}^i(M, R/\mathfrak{M}^n) \to \mathrm{Ext}^i(M, R/\mathfrak{M}^{n-1}),$$

$$\vdots$$

$$(n') \qquad \mathrm{Ext}^i(M, \mathfrak{M}/\mathfrak{M}^2) \to \mathrm{Ext}^i(M, R/\mathfrak{M}^2) \to \mathrm{Ext}^i(M, R/\mathfrak{M}).$$

Each of these Ext modules has finite length since M is finitely generated and all the modules appearing in the second variable have finite length. We now compute these lengths.

First we note that if $X \to Y \to Z$ is a short exact sequence of modules of finite length then

$$\mathrm{length}(Y) \leqq \mathrm{length}(X) + \mathrm{length}(Z).$$

Now let $d_i = [\mathrm{Ext}^i(M, R/\mathfrak{M}) : R/\mathfrak{M}]$. Since $\mathfrak{M}^k/\mathfrak{M}^{k+1} \cong \bigoplus \sum_{e_k} R/\mathfrak{M}$ and Ext commutes with direct sums in either variable,

$$\mathrm{Ext}^i(M, \mathfrak{M}^k/\mathfrak{M}^{k+1}) \cong \bigoplus \sum_{e_k} \mathrm{Ext}^i(M, R/\mathfrak{M}).$$

Hence

$$\mathrm{length}(\mathrm{Ext}^i(M, \mathfrak{M}^k/\mathfrak{M}^{k+1})) = e_k d_i.$$

Thus:

$$(1'') \qquad \mathrm{length}(\mathrm{Ext}^i(M, R/\mathfrak{M}^n)) = e_n d_{i+1},$$

$$(2'') \qquad \mathrm{length}(\mathrm{Ext}^i(M, R/\mathfrak{M}^n)) \leqq e_{n-1} d_i + \mathrm{length}(\mathrm{Ext}^i(M, R/\mathfrak{M}^{n-1})),$$

$$(3'') \qquad \mathrm{length}(\mathrm{Ext}^i(M, R/\mathfrak{M}^{n-1})) \leqq e_{n-2} d_i + \mathrm{length}(\mathrm{Ext}^i(M, R/\mathfrak{M}^{n-2})),$$

$$\vdots$$

$$(n'') \qquad \mathrm{length}(\mathrm{Ext}^i(M, R/\mathfrak{M}^2)) \leqq e_1 d_i + e_0 d_i \qquad (e_0 = 1).$$

Putting these inequalities together: $e_n d_{i+1} \leqq d_i \sum_{k=0}^{n-1} e_k$. Hence $d_{i+1} \leqq (\sum_{k=0}^{n-1} e_k/e_n) d_i = r \cdot d_i$, and by hypothesis, $r<1$. We have, then, $d_2 \leqq r \cdot d_1$, $d_3 \leqq r \cdot d_2, \cdots, d_{i+1} \leqq r \cdot d_i$. Hence for all $i \geqq 1$, $d_{i+1} \leqq r^i d_1$. Since $r<1$, for large enough i, $r^i d_1 < 1$. But d_{i+1} is a nonnegative integer, so $d_{i+1} = 0$. Hence $\mathrm{Ext}^{i+1}(M, R/\mathfrak{M}) = 0$. $R/\mathfrak{M}$ is a strong test module, so pd $M = j < i+1$. Thus for any module N, $\mathrm{Ext}^j(M, N) \neq 0$ [2, Proposition 4.10]. In particular, then, $\mathrm{Ext}^j(M, R/\mathfrak{M}^{n+1}) \neq 0$. By our hypothesis j must be 0, i.e. pd $M = 0$, and since R is local, M is R-free. ∎

COROLLARY 2.3. *Let* $(R, \mathfrak{M})$ *be a local ring with* $\mathfrak{M}^{n+1} = 0$. *If* $e_n > \sum_{i=0}^{n-1} e_i$ *then* R *is a weak test module.*

Now suppose $(R, \mathfrak{M})$ is a regular local ring of dimension k. Let

$\{x_1, \cdots, x_k\}$ be a minimal set of generators for $\mathfrak{M}$. Then the monomials of degree r in $x_1, \cdots, x_k$ form a minimal generating set for $\mathfrak{M}^r$, for any $r \geqq 1$. There are $\binom{k+r-1}{r}$ such monomials, so $e_r = \binom{k+r-1}{r}$. To improve the notation, let us replace e_r with $e_{k,r}$, to indicate explicitly the dependence on k. So $e_{k,0}=1$ and for $r \geqq 1$, $e_{k,r} = \binom{k+r-1}{r}$. Now using induction it is easy to verify that for $r \geqq 1$, $e(k, r)/\sum_{i=0}^{r-1} e(k, i) = k/r$. Hence if $r<k$, $e(k, r) > \sum_{i=0}^{r-1} e(k, i)$. Together with Corollary 2.3 this implies:

Corollary 2.4. *Let $(R, \mathfrak{M})$ be a regular local ring of dimension k. Let $S = R/\mathfrak{M}^{n+1}$, $n+1 \leqq k$. If M is an S-module such that for all $i>0$, $\mathrm{Ext}_S^i(M, S)=0$, then M is S-free. In other words, S, as an S-module, is a weak test module.*

It should be pointed out that if a local ring S, as an S-module, is a weak test module, then every S-module whose Gorenstein dimension (G-dim) is zero is S-free. (Following [1], G-dim $M=0$ if M is reflexive and for all $i>0$, $\mathrm{Ext}_S^i(M, S) = \mathrm{Ext}_S^i(M^*, S) = 0$.) If a local ring S is self-injective, then it is easy to see that every S-module has G-dim zero, so if S is not a field there are nonfree modules M with G-dim $M=0$. However, it is not hard to construct artin local rings which are *not* self-injective and which have nonfree modules whose G-dim is zero.

For example, let $(R, \mathfrak{M})$ be a local Cohen-Macauley ring of dimension one with inj $\dim_R R = \infty$. Let $x \in \mathfrak{M}$ be R-regular and let $S = R/(x^2)$. Then inj $\dim_S S = \infty$, [4, Theorem 2.10], and dim $S=0$, so S is artin. Let $\bar{x}$ = the image of x in S. Then $\mathrm{ann}_S(\bar{x}) = S\bar{x}$. Hence the sequence $0 \to S\bar{x} \to S \to S\bar{x} \to 0$ is exact. Then $(S\bar{x})^* = \mathrm{Hom}_S(S\bar{x}, S) \approx \mathrm{Hom}_S(S/S\bar{x}, S) \approx \mathrm{ann}_S(S\bar{x}) = S\bar{x}$. So $S\bar{x}$ is its own dual (and therefore is reflexive). Thus the above sequence is its own dual and so $\mathrm{Ext}_S^1(S\bar{x}, S)=0$. But $\mathrm{Ext}_S^i(S\bar{x}, S) \approx \mathrm{Ext}_S^{i+1}(S\bar{x}, S)$ for all $i>0$. Hence $\mathrm{Ext}_S^i(S\bar{x}, S)=0$ for all $i>0$. Since $S\bar{x} \approx (S\bar{x})^*$, $\mathrm{Ext}_S^i(S\bar{x}^*, S)=0$ for all $i>0$. So G-dim $S\bar{x}=0$, and $S\bar{x}$ is not free since $\mathrm{ann}_S(S\bar{x}) = S\bar{x} \neq 0$.

We conclude this paper with an observation that the self-injectivity of an artin local ring depends upon the vanishing of a single particular Ext.

Proposition 2.5. *Suppose $(R, \mathfrak{M})$ is an artin local ring with $n+1$ = index of nilpotence of $\mathfrak{M}$. Then R is self-injective $\Leftrightarrow \mathrm{Ext}^1(R/\mathfrak{M}^n, R)=0$.*

Proof. ($\Rightarrow$) Obvious.

($\Leftarrow$) Dualizing the exact sequence $0 \to \mathfrak{M}^n \to R \to R/\mathfrak{M}^n \to 0$ we have that

$$0 \to \operatorname{Hom}(R/\mathfrak{M}^n, R) \to R \to \operatorname{Hom}(\mathfrak{M}^n, R) \to 0$$

is exact. $\operatorname{Hom}(R/\mathfrak{M}^n, R) \approx \operatorname{ann}(\mathfrak{M}^n) = \mathfrak{M}$ (since $\mathfrak{M}^{n+1} = 0$ and $\mathfrak{M}^n \neq 0$). Hence $\operatorname{Hom}(\mathfrak{M}^n, R) \approx R/\mathfrak{M}$. But $\mathfrak{M}^n$ is an $R/\mathfrak{M}$-module and thus a direct sum of copies of $R/\mathfrak{M}$. Therefore $\operatorname{Hom}(\mathfrak{M}^n, R)$ is a direct sum of copies of $\operatorname{Hom}(R/\mathfrak{M}, R)$. Since, on the other hand, it is a simple module, we conclude that $R/\mathfrak{M} \approx \operatorname{Hom}(\mathfrak{M}^n, R) \approx \operatorname{Hom}(R/\mathfrak{M}, R)$. Now $\operatorname{Hom}(R/\mathfrak{M}, R)$ is the socle of R, and it is well known (see, for example, [**3**, Corollary 2.8]) that R is self-injective if and only if length(socle of R) $= 1$, so we are done. ∎

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