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A NOTE ON THE KLEINECKE-SHIROKOV THEOREM AND THE WINTNER-WIELANDT-HALMOS THEOREM

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ABSTRACT. We extend the Kleinecke-Shirokov theorem to the almost commutative case. From this result we prove the Wintner-Wielandt-Halmos theorem.

The Kleinecke-Shirokov theorem says, according to [1, p. 128], that if P and Q are bounded linear operators on a Hilbert space, $T=PQ-QP$ and T commutes with P, then T is quasinilpotent.

THEOREM 1. *If P and Q are bounded linear operators on a Banach space X, $T=PQ-QP$ and T almost commutes with P, i.e., $PT-TP$ is a compact operator, then T is a Riesz operator. In particular, if X is a Hilbert space, then there exists a compact normal operator S such that $T-S$ is a quasinilpotent operator.*

PROOF. We can prove the theorem by an analogous method to Kleinecke's original one [1, p. 335]. Let c be the canonical homomorphism of $B(X)$, the Banach algebra of bounded linear operators on X, onto $B(X)/K(X)$, the quotient Banach algebra of $B(X)$ modulo $K(X)$, the closed two-sided ideal of compact operators on X. Let $c(P)$ be fixed and $D(c(Q))=c(P)c(Q)-c(Q)c(P)$ be a function of $c(Q)$, then D is a bounded linear operator on $B(X)/K(X)$ which is also a derivation. By using the Leibniz formula and the fact that $D^2(c(Q))=D(c(T))=c(P)c(T)-c(T)c(P)=0$, it can be shown that $D^n(c(Q)^n)=n!(D(c(Q)))^n=n!c(T)^n$. Hence $\|c(T)^n\|^{1/n}\to 0$ as $n\to\infty$, i.e., T is a Riesz operator [2]. The particular case will follow from the following proposition: If we denote by $R(X)$ the set of Riesz operators, $N(X)$ the set of quasinilpotent operators and $KM(X)$ the set of compact normal operators on X, then

$$R(X) = N(X) + KM(X) = N(X) + K(X) = R(X) + K(X).$$

In fact, every Riesz operator on a Hilbert space is decomposable as the sum of a quasinilpotent operator and a compact normal operator [3, Theorem 7.5]. Hence $R(X)\subseteq N(X)+KM(X)\subseteq N(X)+K(X)\subseteq R(X)+K(X)$. On the other hand, if $T\in R(X)$ and $S\in K(X)$, then

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$c(T+S)=c(T)$ is quasinilpotent, $T+S\in R(X)$ and hence $R(X)+K(X)\subseteq R(X)$.

THEOREM 2 (WINTNER-WIELANDT-HALMOS THEOREM). *If b is a nonzero scalar and S is a compact operator on a Banach space X, then $b+S$ is not a commutator.*

PROOF. Suppose on the contrary that $b+S=PQ-QP$ with P and $Q\in B(X)$, then $P(b+S)-(b+S)P=PS-SP\in K(X)$. Hence $b+S\in R(X)$ by Theorem 1. But $-S\in R(X)$ and $(-S)(b+S)=(b+S)(-S)$, thus $b=(b+S)-S\in R(X)$ [**2**, Theorem 3.1]. By the definition of a Riesz operator, this is impossible unless $b=0$. This shows that $b+S$ is not a commutator.

If S is the zero operator, Theorem 2 is precisely the Wintner-Wielandt theorem, i.e., the only scalar commutator is 0.

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