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# KRONECKER FUNCTION RINGS AND FLAT $D[X]$-MODULES[1]

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ABSTRACT. Let $D$ be an integral domain with identity. Gilmer has recently shown that in order that a $v$-domain $D$ be a Prüfer $v$-multiplication ring, it is necessary and sufficient that $D^v$ be a quotient ring of $D[X]$, where $D^v$ is the Kronecker function ring of $D$ with respect to the $v$-operation. In this paper the authors prove that in the above theorem it is possible to replace "a quotient ring of $D[X]$" with "a flat $D[X]$-module." Moreover, it is shown that $D^v$ is the only Kronecker function ring of $D[X]$ which can ever be a flat $D[X]$-module.

In the sequel $D$ will denote an integral domain with identity and $K$ will denote its quotient field. Otherwise, our notation is essentially that of [1].

Let $I(D)$ denote the collection of all fractional ideals of $D$. The mapping $F \to F_v$ of $I(D)$ into $I(D)$, where $F_v = (F^{-1})^{-1}$, is called the $v$-operation on $D$. The $v$-operation satisfies the properties of a *-operation, and if the $v$-operation is endlich arithmetisch brauchbar, then we call $D$ a *$v$-domain* and we denote by $D^v$ the Kronecker function ring of $D$ with respect to the $v$-operation (for a detailed treatment of *-operations, Kronecker function rings and the $v$-operation, see [1, Chapters 26 and 28]). If the set of $v$-ideals of finite type is a group under $v$-multiplication, then $D$ is said to be a *Prüfer $v$-multiplication ring*. Let $D$ be a $v$-domain. Then in [2] Gilmer proves that $D$ is a Prüfer $v$-multiplication ring if and only if $D^v$ is a quotient ring of $D[X]$. Thus, in case $D$ is a Prüfer $v$-multiplication ring, $D^v$ is a flat $D[X]$-module. The converse is also true, but in order to prove it we require some preliminary results.

LEMMA 1. *Let $D$ be an integral domain. If $Q$ is a prime ideal of $D[X]$ such that $(D[X])_Q$ is a valuation ring and if $(Q\cap D)D[X]\subset Q$, then $Q\cap D=(0)$.*

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PROOF. Since $D_{Q\cap D}=(D[X])_Q\cap K$, there is no loss of generality in assuming that $D$ is a valuation ring and that $Q\cap D$ is its maximal ideal. $Q$ is generated mod $(Q\cap D)D[X]$ by a monic polynomial $f$. If $y\in Q\cap D$, then since $(D[X])_Q$ is a valuation ring and since $f\notin (Q\cap D)(D[X])_Q$, it follows that $y=fg/h$, where $g\in D[X]$ and $h\in D[X]-Q$. If $y\neq 0$, then $f$ divides $h$ in $K[X]$, whence $f$ divides $h$ in $D[X]$ by [1, 8.4].

The following result, due to Richman [3, Theorem 2], will be of use.

LEMMA 2. *Let $D_1$ be an overring of $D$—that is, $D\subseteq D_1\subseteq K$. In order that $D_1$ be a flat $D$-module it is necessary and sufficient that $(D_1)_{M_1}=D_{M_1\cap D}$ for each maximal ideal $M_1$ of $D_1$.*

We are now able to sharpen the aforementioned result of Gilmer.

THEOREM 3. *Let $D$ be a $v$-domain and let $D^v$ be the Kronecker function ring of $D$ with respect to the $v$-operation. The following conditions are equivalent:*

(1) *$D$ is a Prüfer $v$-multiplication ring.*

(2) *$D^v$ is a quotient ring of $D[X]$.*

(3) *Each valuation overring of $D^v$ is of the form $(D[X])_{P[X]}$ where $D_P$ is a valuation overring of $D$.*

(4) *$D^v$ is a flat $D[X]$-module.*

PROOF. The equivalence of (1) and (2) is given in [2]. That (2) implies (3) is a direct consequence of Lemma 1 and that (3) implies (4) follows from Lemma 2. Therefore, we need only show that (4) implies (2). We claim that $D^v=(D[X])_S$, where $S=\{f\in D[X] \mid (A_f)_v=D\}$. (Here, $A_f$ denotes the ideal of $D$ generated by the coefficients of $f$.) Clearly, $D^v\supseteq(D[X])_S$. Let $A$ be an ideal of $D[X]$ such that $AD^v=D^v$. Then there exist $f_1, \cdots, f_n\in A$ such that $(f_1, \cdots, f_n)D^v=D^v$. Set $m=\max_{1\leq i\leq n}\{\deg(f_i)\}+1$ and put $f(X)=f_1+f_2X^m+\cdots+f_nX^{(n-1)m}$. Then by [1, 26.7], $(A_f)_v=D$ and hence $A\cap S\neq\varnothing$. Therefore, if $M$ is a maximal ideal of $(D[X])_S$, then $MD^v\subset D^v$ and there exists a maximal ideal $M'$ of $D^v$ such that $M=M'\cap(D[X])_S$. The result follows from Lemma 2.

On the basis of Theorem 3, one is led to ask what Kronecker function rings are flat $D[X]$-modules. The answer is given by

COROLLARY 4. *If $D'$ is a Kronecker function ring of $D$ which is a flat $D[X]$-module, then $D'=D^v$.*

PROOF. It follows from Lemma 1 and Lemma 2 that each valuation overring of $D'$ is of the form $(D[X])_{P_\alpha[X]}$, where $D_{P_\alpha}$ is a valuation overring of $D$. Therefore, $D=D'\cap K=(\bigcap_\alpha(D[X])_{P_\alpha[X]})\cap K=\bigcap_\alpha D_{P_\alpha}$ and it follows from [1, 36.13] that $D'=D^v$.

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